2

Dynamics of Ideal Fluids

The basic goal of any fluid-dynamical study is to provide (1) a complete description of the *motion* of the fluid at any instant of time, and hence of the *kinematics* of the flow, and (2) a description of how the motion changes in time in response to applied forces, and hence of the *dynamics* of the flow. We begin our study of astrophysical fluid dynamics by analyzing the motion of a compressible *ideal fluid* (i.e., a nonviscous and nonconducting gas); this allows us to formulate very simply both the basic conservation laws for the mass, momentum, and energy of a fluid parcel (which govern its dynamics) and the essentially geometrical relationships that specify its kinematics. Because we are concerned here with the macroscopic properties of the flow of a physically uncomplicated medium, it is both natural and advantageous to adopt a purely continuum point of view. In the next chapter, where we seek to understand the important role played by internal processes of the gas in transporting energy and momentum within the fluid, we must carry out a deeper analysis based on a kinetic-theory view; even then we shall see that the continuum approach yields useful results and insights. We pursue this line of inquiry even further in Chapters 6 and 7, where we extend the analysis to include the interaction between radiation and both the internal state, and the macroscopic dynamics, of the material.

2.1 Kinematics

15. Velocity and Acceleration

In developing descriptions of fluid motion it is fruitful to work in two different frames of reference, each of which has distinct advantages in certain situations. On the one hand we can view the flow from a fixed *laboratory frame*, and consider any property of the fluid, say α, to be a function of the position $\mathbf{x}$ in this frame and of time t, that is, $\alpha = \alpha(\mathbf{x}, t)$. The time and space variation of α are then described using a time derivative $(\partial/\partial t)$ computed at fixed $\mathbf{x}$ and space derivatives $(\partial/\partial x^i)$ evaluated at fixed t. This approach is generally known as the *Eulerian description*. In particular, in this scheme we describe the *velocity* of the fluid by a vector field $\mathbf{v}(\mathbf{x}, t)$, which gives the rate and direction of flow of the material as a function of position and time, as seen in the laboratory frame.

Alternatively, we may choose a particular fluid parcel and study the time variation of its properties while following the motion of that parcel; this approach is generally referred to as the *Lagrangean description.* The time variation of the properties of a Lagrangean fluid element (also called a *material element*) is described in terms of the *fluid-frame time derivative* (D/Dt) (also known as the *comoving*, or *Lagrangean*, or *material*, or *substantial derivative*). For example, the velocity of an element of fluid is, by definition, the time rate of change of the position of that particular fluid parcel; hence in the Lagrangean scheme we have

$$\mathbf{v} = (D\mathbf{x}/Dt). \qquad (15.1)$$

Similarly the *acceleration* **a** of a fluid element is the rate of change of its velocity during the course of its motion, hence

$$\mathbf{a} = (D\mathbf{v}/Dt). \qquad (15.2)$$

For nonrelativistic flows, the choice between the Eulerian or Lagrangean points of view is usually made purely on the basis of the convenience of one or the other frame for formulating the physics of the situation under study; as we show below, derivatives in the two frames are simply related. In relativistic flows, however, the difference between these two frames is much more fundamental and has a deep physical significance; we return to this point in Chapter 4.

To relate the Lagrangean time derivative to derivatives in the Eulerian frame, we notice that $(D\alpha/Dt)$ is defined as

$$(D\alpha/Dt) = \lim_{\Delta t \to 0} [\alpha(\mathbf{x}+\Delta\mathbf{x}, t+\Delta t) - \alpha(\mathbf{x}, t)]/\Delta t \qquad (15.3)$$

where α is measured in a definite fluid parcel at two different times, t and $t+\Delta t$, and also, as a result of the motion of that parcel, at two different positions, $\mathbf{x}$ and $\mathbf{x}+\Delta\mathbf{x} = \mathbf{x}+\mathbf{v}\,\Delta t$, as seen in the laboratory frame. Expanding to first order in Δt, we have

$$\begin{aligned}\alpha(\mathbf{x}+\Delta\mathbf{x}, t+\Delta t) &= \alpha(\mathbf{x}, t) + (\partial\alpha/\partial t)\,\Delta t + (\partial\alpha/\partial x^i)\,\Delta x^i \\ &= \alpha(\mathbf{x}, t) + [(\partial\alpha/\partial t) + v^i(\partial\alpha/\partial x^i)]\,\Delta t \qquad (15.4) \\ &= \alpha(\mathbf{x}, t) + (\alpha_{,t} + v^i\alpha_{,i})\,\Delta t,\end{aligned}$$

(cf. §A1 for notation) and therefore

$$(D\alpha/Dt) = \alpha_{,t} + v^i\alpha_{,i}. \qquad (15.5)$$

Equation (15.5) holds for any α: scalar, vector, or tensor. In more familiar notation,

$$(D\alpha/Dt) = (\partial\alpha/\partial t) + (\mathbf{v}\cdot\boldsymbol{\nabla})\alpha. \qquad (15.6)$$

The covariant generalization of (15.5), valid in curvilinear coordinates, is

$$(D\alpha/Dt) = \alpha_{,t} + v^i\alpha_{;i} \qquad (15.7)$$

where $\alpha_{;i}$ is the *covariant derivative* of α with respect to x^i (cf. §A3.10). By comparing (15.7) with equation (A3.81) we see that, in general, $(D\alpha/Dt)$ is to be identified with the *intrinsic derivative* $(\delta\alpha/\delta t)$, a remark that will assume greater significance in our discussion of relativistic kinematics. It should be noted that here, and in Chapter 3, by the "covariant form" of an equation we mean covariant only with respect to changes of the spatial coordinates in a three-dimensional Euclidian space, treating time as absolute and independent. In Chapters 4 and 7 we use this expression to mean completely covariant with respect to the full four-dimensional spacetime of special relativity.

The value of the formalism provided by (15.7) is seen when we derive vector and tensor expressions in curvilinear coordinates. Consider, for example, the acceleration, which in Cartesian coordinates is simply

$$a^i = v^i_{,t} + v^j v^i_{,j}. \tag{15.8}$$

In curvilinear coordinates we have, from equation (A3.83),

$$a^i = v^i_{,t} + v^j v^i_{;j} = v^i_{,t} + v^j v^i_{,j} + \begin{Bmatrix} i \\ j \;\; k \end{Bmatrix} v^j v^k. \tag{15.9}$$

In particular, for spherical coordinates we find from equation (A3.63) that the contravariant components of **a** are

$$a^{(1)} = v^{(1)}_{,t} + v^j v^{(1)}_{,j} - r(v^{(2)})^2 - r \sin^2\theta (v^{(3)})^2, \tag{15.10a}$$

$$a^{(2)} = v^{(2)}_{,t} + v^j v^{(2)}_{,j} + 2v^{(1)}v^{(2)}/r - \sin\theta\cos\theta (v^{(3)})^2, \tag{15.10b}$$

and

$$a^{(3)} = v^{(3)}_{,t} + v^j v^{(3)}_{,j} + 2v^{(1)}v^{(3)}/r + 2\cot\theta\, v^{(2)}v^{(3)}. \tag{15.10c}$$

Now replacing the contravariant components of both **a** and **v** with their equivalent physical components via equation (A3.46a), we find

$$a_r = \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{1}{r}(v_\theta^2 + v_\phi^2), \tag{15.11a}$$

$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{\cot\theta}{r} v_\phi^2, \tag{15.11b}$$

$$a_\phi = \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{\cot\theta}{r} v_\theta v_\phi. \tag{15.11c}$$

Similar expressions for any other coordinate system are easily derived from (15.9).

16. Particle Paths, Streamlines, and Streaklines

In a moving fluid, the *particle path* of a particular fluid element is simply the three-dimensional path traced out in time by that element. If we label each element by its coordinates $\boldsymbol{\xi}$ at some reference time $t = 0$, then its particle path is

$$\mathbf{x}(\boldsymbol{\xi}, t) = \boldsymbol{\xi} + \int_0^t \mathbf{v}(\boldsymbol{\xi}, t')\, dt' \tag{16.1}$$

where $\mathbf{v}(\boldsymbol{\xi}, t)$ is the velocity, as a function of time, of the element specified by $\boldsymbol{\xi}$. Equation (16.1) is the parametric equation of a curve in space, with t as parameter.

The *streamlines* in a flow are defined as those curves that, at a *given instant of time*, are tangent at each point to the velocity of the fluid at that position (and time). Thus a streamline can be written as the parametric curve

$$(d\mathbf{x}/ds) = \mathbf{v}(\mathbf{x}, t) \tag{16.2}$$

where t is fixed and s is a path-length parameter. Alternatively, along a streamline,

$$(dx/v_x) = (dy/v_y) = (dz/v_z) = ds. \tag{16.3}$$

Notice also that if $d\mathbf{x}$ lies along a streamline, then $\mathbf{v} \times d\mathbf{x} = 0$. A *stream tube* is a tube, filled with flowing fluid, whose surface is composed of all the streamlines passing through some closed curve C.

A *streakline* is the curve traced out in time by all fluid particles that pass through a given fixed point in the flow field; streaklines may be made visible in a flow, for example, by injection of a dye or colored smoke at some point in the flow.

For *steady* flows, where $\mathbf{v}(\mathbf{x}, t)$ is independent of time, the particle paths, streamlines, and streaklines are all identical; in time-dependent flows all three sets of curves will, in general, be distinct.

17. *The Euler Expansion Formula*

Suppose we choose a fluid element located, at $t = 0$, within the volume element $dV_0 = d\xi^{(1)}\, d\xi^{(2)}\, d\xi^{(3)}$ around some point $\boldsymbol{\xi}$. As the fluid flows, in time this element will, in general, move to some other position $\mathbf{x}(\boldsymbol{\xi}, t)$, and will occupy some new volume element $dV = dx^{(1)}\, dx^{(2)}\, dx^{(3)}$. These volume elements are related by the expression

$$\begin{aligned} dV &= J(x^{(1)}, x^{(2)}, x^{(3)}/\xi^{(1)}, \xi^{(2)}, \xi^{(3)})\, d\xi^{(1)}\, d\xi^{(2)}\, d\xi^{(3)} \\ &= J\, d\xi^{(1)}\, d\xi^{(2)}\, d\xi^{(3)} = J\, dV_0, \end{aligned} \tag{17.1}$$

where J is the Jacobian of the transformation $\mathbf{x}(\boldsymbol{\xi}, t)$. The ratio

$$J = (dV/dV_0) \tag{17.2}$$

is called the *expansion* of the fluid.

In what follows, we require an expression for the time rate of change of the expansion of a fluid element as we follow its motion in the flow. From equation (A2.23),

$$J=\left|\frac{\partial x^i}{\partial \xi^j}\right|=e^{ijk}\frac{\partial x^{(1)}}{\partial \xi^i}\frac{\partial x^{(2)}}{\partial \xi^j}\frac{\partial x^{(3)}}{\partial \xi^k}, \tag{17.3}$$

hence

$$\begin{aligned}\frac{DJ}{Dt}&=\frac{D}{Dt}\left(e^{ijk}\frac{\partial x^{(1)}}{\partial \xi^i}\frac{\partial x^{(2)}}{\partial \xi^j}\frac{\partial x^{(3)}}{\partial \xi^k}\right)\\&=e^{ijk}\frac{\partial v^{(1)}}{\partial \xi^i}\frac{\partial x^{(2)}}{\partial \xi^j}\frac{\partial x^{(3)}}{\partial \xi^k}+e^{ijk}\frac{\partial x^{(1)}}{\partial \xi^i}\frac{\partial v^{(2)}}{\partial \xi^j}\frac{\partial x^{(3)}}{\partial \xi^k}\\&\quad+e^{ijk}\frac{\partial x^{(1)}}{\partial \xi^i}\frac{\partial x^{(2)}}{\partial \xi^j}\frac{\partial v^{(3)}}{\partial \xi^k}.\end{aligned} \tag{17.4}$$

We can expand $(\partial v^i/\partial \xi^j)$ as

$$(\partial v^i/\partial \xi^j)=(\partial v^i/\partial x^l)(\partial x^l/\partial \xi^j). \tag{17.5}$$

Therefore,

$$\begin{aligned}\frac{DJ}{Dt}&=e^{ijk}\frac{\partial v^{(1)}}{\partial x^l}\frac{\partial x^l}{\partial \xi^i}\frac{\partial x^{(2)}}{\partial \xi^j}\frac{\partial x^{(3)}}{\partial \xi^k}+e^{ijk}\frac{\partial x^{(1)}}{\partial \xi^i}\frac{\partial v^{(2)}}{\partial x^l}\frac{\partial x^l}{\partial \xi^j}\frac{\partial x^{(3)}}{\partial \xi^k}\\&\quad+e^{ijk}\frac{\partial x^{(1)}}{\partial \xi^i}\frac{\partial x^{(2)}}{\partial \xi^j}\frac{\partial v^{(3)}}{\partial x^l}\frac{\partial x^l}{\partial \xi^k}.\end{aligned} \tag{17.6}$$

Writing out the first term on the right-hand side in full, we have

$$\begin{aligned}e^{ijk}\frac{\partial v^{(1)}}{\partial x^l}\frac{\partial x^l}{\partial \xi^i}\frac{\partial x^{(2)}}{\partial \xi^j}\frac{\partial x^{(3)}}{\partial \xi^k}&=\frac{\partial v^{(1)}}{\partial x^{(1)}}e^{ijk}\frac{\partial x^{(1)}}{\partial \xi^i}\frac{\partial x^{(2)}}{\partial \xi^j}\frac{\partial x^{(3)}}{\partial \xi^k}\\&\quad+\frac{\partial v^{(1)}}{\partial x^{(2)}}e^{ijk}\frac{\partial x^{(2)}}{\partial \xi^i}\frac{\partial x^{(2)}}{\partial \xi^j}\frac{\partial x^{(3)}}{\partial \xi^k}\\&\quad+\frac{\partial v^{(1)}}{\partial x^{(3)}}e^{ijk}\frac{\partial x^{(3)}}{\partial \xi^i}\frac{\partial x^{(2)}}{\partial \xi^j}\frac{\partial x^{(3)}}{\partial \xi^k}.\end{aligned} \tag{17.7}$$

The last two terms on the right-hand side of (17.7) are obviously identically zero because the permutation symbol is antisymmetric in all indices, while the first equals $(\partial v^{(1)}/\partial x^{(1)})J$. By expanding the second and third terms on the right-hand side of (17.6) in a similar way, we obtain finally

$$\frac{DJ}{Dt}=\left(\frac{\partial v^{(1)}}{\partial x^{(1)}}+\frac{\partial v^{(2)}}{\partial x^{(2)}}+\frac{\partial v^{(3)}}{\partial x^{(3)}}\right)J=(\nabla\cdot\mathbf{v})J, \tag{17.8}$$

which is known as *Euler's expansion formula.*

The covariant generalization of (17.8) to curvilinear coordinates, taking account of equation (A3.86), is

$$(D\ln J/Dt)=v^i_{;i}=g^{-1/2}(g^{1/2}v^i)_{,i} \tag{17.9}$$

where g is the determinant of the metric tensor of the coordinate system (cf. §A3.4).

18. The Reynolds Transport Theorem

With the help of Euler's expansion formula, we can now calculate the time rate of change of integrals of physical quantities within material volumes. Thus let $F(\mathbf{x}, t)$ be any single-valued scalar, vector, or tensor field, and choose $\mathcal{V}(t)$ to be some finite *material* volume composed of a definite set of fluid particles. Then, clearly,

$$\mathcal{F}(t)=\int_{\mathcal{V}} F(\mathbf{x}, t)\, dV \tag{18.1}$$

is a definite function of time. At $t=0$, the material element dV is identical to a *fixed* element dV_0; at later times dV is related to dV_0 by means of (17.8). The fluid-frame time derivative of $\mathcal{F}$ is then

$$\begin{aligned}\frac{D\mathcal{F}}{Dt}=\frac{D}{Dt}\int_{V_0} F(\mathbf{x}, t)J\, dV_0 &= \int_{V_0}\left(\frac{DF}{Dt}J+F\frac{DJ}{Dt}\right) dV_0\\ &= \int_{V_0}\left(\frac{DF}{Dt}+F\,\boldsymbol{\nabla}\cdot\mathbf{v}\right)J\, dV_0,\end{aligned}$$

hence

$$\frac{D\mathcal{F}}{Dt}=\int_{\mathcal{V}}\left(\frac{DF}{Dt}+F\,\boldsymbol{\nabla}\cdot\mathbf{v}\right) dV. \tag{18.2}$$

In view of (15.6) and equation (A2.48) we can also write

$$\frac{D}{Dt}\int_{\mathcal{V}} F(\mathbf{x}, t)\, dV=\int_{\mathcal{V}}\left[\frac{\partial F}{\partial t}+\boldsymbol{\nabla}\cdot(F\mathbf{v})\right] dV. \tag{18.3}$$

Furthermore, by applying the divergence theorem [cf. (A2.68)], we also have

$$\frac{D}{Dt}\int_{\mathcal{V}} F(\mathbf{x}, t)\, dV=\int_{\mathcal{V}} \frac{\partial F}{\partial t}\, dV+\int_{S} F\mathbf{v}\cdot d\mathbf{S} \tag{18.4}$$

where S is the surface bounding $\mathcal{V}$.

Equations (18.2) through (18.4) are known as the *Reynolds transport theorem*; they hold for any scalar field or the components of a vector or tensor field. We shall use these important results repeatedly in what follows. The physical interpretation of (18.4) is quite clear: the rate of change of the integral of F within a material volume equals the integral of the time rate of change of F within the fixed volume V that instantaneously coincides with the material volume $\mathcal{V}(t)$, plus the net flux of F through the bounding surface of the (moving) material volume.

19. The Equation of Continuity

By definition the fluid in a material volume is always composed of exactly the same particles. The mass contained within a material volume must therefore always be the same; hence

$$\frac{D}{Dt}\int_{\mathcal{V}} \rho\, dV \equiv 0. \tag{19.1}$$

Equation (19.1) is a mathematical statement of the law of *conservation of mass* for the fluid. Now applying the Reynolds theorem (18.2) for $F=\rho$, we find

$$\int_{\mathcal{V}} [(D\rho/Dt)+\rho(\nabla \cdot \mathbf{v})]\, dV = 0. \tag{19.2}$$

But the material volume $\mathcal{V}$ is arbitrary, so in general we can guarantee that the integral will vanish only if the integrand vanishes at all points in the flow field. We thus obtain the *equation of continuity*

$$(D\rho/Dt)+\rho(\nabla \cdot \mathbf{v}) = 0, \tag{19.3}$$

or, from (15.6),

$$(\partial\rho/\partial t)+\nabla \cdot (\rho\mathbf{v}) = \rho_{,t}+(\rho v^i)_{,i} = 0. \tag{19.4}$$

For *steady flow* $(\partial\rho/\partial t)\equiv 0$, hence

$$\nabla \cdot (\rho\mathbf{v}) = 0. \tag{19.5}$$

For a one-dimensional steady flow in planar geometry we then have $d(\rho v_z)/dz = 0$ or

$$\rho v_z = \text{constant} = \dot{m}, \tag{19.6}$$

where $\dot{m}$ is the *mass flux* per unit area in the flow.

For *incompressible flow* $\rho \equiv$constant, hence $(\partial\rho/\partial t)\equiv 0$ and from (19.5) $\nabla \cdot \mathbf{v} \equiv 0$.

The covariant generalization of (19.4), valid in curvilinear coordinate systems, is

$$\rho_{,t}+(\rho v^i)_{;i} = 0. \tag{19.7}$$

For example, in spherical coordinates [for which $g^{1/2}=r^2 \sin\theta$ and the connection between contravariant and physical components is given by equation (A3.46a)], we find, by using equation (A3.86), that

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho v_r)+\frac{1}{r\sin\theta}\frac{\partial(\rho v_\theta \sin\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial(\rho v_\phi)}{\partial\phi} = 0. \tag{19.8}$$

For one-dimensional, spherically symmetric flow, (19.8) simplifies to

$$(\partial\rho/\partial t)+r^{-2}[\partial(r^2\rho v_r)/\partial r] = 0, \tag{19.9}$$

which, for steady flow, implies that

$$4\pi r^2 \rho v_r = \text{constant} = \dot{\mathcal{M}}, \tag{19.10}$$

where $\dot{\mathcal{M}}$ is the mass flux through a spherical shell surrounding the origin.

Finally, by applying the Reynolds transport theorem (18.2) to the function $F=\rho\alpha$, we see that

$$\frac{D}{Dt}\int_{\mathcal{V}}\rho\alpha\,dV=\int_{\mathcal{V}}\left[\frac{D(\rho\alpha)}{Dt}+\rho\alpha(\nabla\cdot\mathbf{v})\right]dV$$

$$=\int_{\mathcal{V}}\left\{\rho\frac{D\alpha}{Dt}+\alpha\left[\frac{D\rho}{Dt}+\rho(\nabla\cdot\mathbf{v})\right]\right\}dV, \quad (19.11)$$

which, in view of the equation of continuity (19.3), yields the useful identity

$$\frac{D}{Dt}\int_{\mathcal{V}}\rho\alpha\,dV=\int_{\mathcal{V}}\rho\frac{D\alpha}{Dt}\,dV. \quad (19.12)$$

Similarly, by combining (19.12) with (18.3), we find the useful result

$$\rho(D\alpha/Dt)=[\partial(\rho\alpha)/\partial t]+\nabla\cdot(\rho\alpha\mathbf{v}). \quad (19.13)$$

In (19.12) and (19.13), α may be a scalar or a component of a vector or tensor.

20. *Vorticity and Circulation*

The *vorticity* $\boldsymbol{\omega}$ of a fluid flow is defined to be the curl of the velocity field:

$$\boldsymbol{\omega}\equiv\nabla\times\mathbf{v}. \quad (20.1)$$

In Cartesian coordinates

$$\omega^i=e^{ijk}v_{k,j}. \quad (20.2)$$

We shall see in §21 that $\boldsymbol{\omega}$ provides a local measure of the rate of rotation of the fluid at each point in the flow. Therefore we define an *irrotational flow* to be one for which $\boldsymbol{\omega}\equiv 0$. Such flows can be described by a scalar *velocity potential* ϕ, defined to be such that $\mathbf{v}=\nabla\phi$, because then by equation (A2.56) we have $\boldsymbol{\omega}=\nabla\times(\nabla\phi)\equiv 0$.

A *vortex line* is a curve that at each point is tangent to the vortex vector at that point. Thus if $d\mathbf{x}$ lies along a vortex line we have

$$(dx/\omega_x)=(dy/\omega_y)=(dz/\omega_z)=ds, \quad (20.3)$$

which generates a parametric curve with s as the parameter. Clearly we must also have $\boldsymbol{\omega}\times d\mathbf{x}=0$ along a vortex line. A *vortex tube* is the surface generated by all vortex lines passing through some closed curve C in the fluid.

The *circulation* in the flow is defined to be

$$\Gamma\equiv\oint_C\mathbf{v}\cdot d\mathbf{x} \quad (20.4)$$

where C is a closed curve. Suppose, in fact, that C is a closed material curve, that is, a closed curve composed of a definite set of fluid particles.

Then the time rate of change of the circulation around that material curve is

$$\frac{D\Gamma}{Dt}=\frac{D}{Dt}\oint_C \mathbf{v}\cdot d\mathbf{x}=\oint_C \frac{D\mathbf{v}}{Dt}\cdot d\mathbf{x}+\oint_C \mathbf{v}\cdot\frac{D}{Dt}(d\mathbf{x}). \tag{20.5}$$

But, by (15.1), on a material curve we have $[D(d\mathbf{x})/Dt]=d(D\mathbf{x}/Dt)=d\mathbf{v}$, hence

$$(D\Gamma/Dt)=\oint_C (D\mathbf{v}/Dt)\cdot d\mathbf{x}+\oint_C \mathbf{v}\cdot d\mathbf{v}=\oint_C (D\mathbf{v}/Dt)\cdot d\mathbf{x}+\oint_C d(\tfrac{1}{2}v^2). \tag{20.6}$$

However v^2 is a single-valued scalar function, and its line integral around a closed path must be identically zero. Hence we obtain *Kelvin's equation*

$$(D\Gamma/Dt)=\oint_C (D\mathbf{v}/Dt)\cdot d\mathbf{x}. \tag{20.7}$$

The *strength* of a vortex tube (or the *flux of vorticity*) is defined to be

$$\Sigma=\int_S \boldsymbol{\omega}\cdot d\mathbf{S} \tag{20.8}$$

where S is the cross section enclosed by a closed curve C lying in the surface of the tube. By Stokes's theorem (A2.70), we then see that the strength of a vortex tube equals the circulation in the flow:

$$\Sigma=\int_S (\nabla\times\mathbf{v})\cdot d\mathbf{S}=\oint_C \mathbf{v}\cdot d\mathbf{x}=\Gamma. \tag{20.9}$$

Because the divergence of the vorticity is identically zero [cf. (A2.54)], it follows from the divergence theorem (A2.68) that the flux of vorticity $\boldsymbol{\omega}\cdot d\mathbf{S}$ integrated over a closed surface S, which is composed of two cross sections S_1 and S_2 of a vortex tube (bounded by closed curves C_1 and C_2) and the segment of the tube joining them, must be zero. Furthermore, because $\boldsymbol{\omega}$ by definition lies along the tube, it is obvious that $\boldsymbol{\omega}\cdot d\mathbf{S}$ must be identically zero on that part of the closed surface. We must therefore have

$$\int_{S_1} (\nabla\times\mathbf{v})\cdot\mathbf{n}_1\,dS+\int_{S_2} (\nabla\times\mathbf{v})\cdot\mathbf{n}_2\,dS=\Sigma_1+\Sigma_2=\Gamma_1-\Gamma_2=0, \tag{20.10}$$

where in choosing the sign of Γ_2 we have recognized that $\mathbf{n}_2$, the outward normal on S_2, is opposite in direction to the positive normal (in the sense of being right handed) defined by the circuit of C_2. Thus we have *Helmholtz's vortex theorem*: the flux of vorticity across any cross-section of a vortex tube is constant, or, equivalently, the circulation around any closed surface lying on the surface of a vortex tube is constant. It should be noted explicitly that the fluid particles that define the surface of a vortex tube at one time will not in general lie on its surface at a later time. Thus (20.10) does not automatically imply that Γ around a *material* curve is constant; we return to the question of when $(D\Gamma/Dt)=0$ in §23.

The covariant generalization of (20.2) is

$$\omega^i = \varepsilon^{ijk} v_{k;j} = g^{-1/2}(v_{k,j} - v_{j,k}), \tag{20.11}$$

where ε^{ijk} is the Levi-Civita tensor, and the second equality is proved in equation (A3.95). For example, in spherical coordinates we find from equation (A3.96)

$$\omega_r = \frac{1}{r\sin\theta}\left[\frac{\partial(v_\phi \sin\theta)}{\partial\theta} - \frac{\partial v_\theta}{\partial\phi}\right], \tag{20.12a}$$

$$\omega_\theta = \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial(r v_\phi)}{\partial r}\right], \tag{20.12b}$$

and

$$\omega_\phi = \frac{1}{r}\left[\frac{\partial(r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial\theta}\right]. \tag{20.12c}$$

Finally, we note that the vorticity $\boldsymbol{\omega}$ is in reality a pseudovector, and has associated with it (cf. §A2.11) an antisymmetric second-rank tensor

$$\Omega_{ij} = e_{ijk}\omega^k = v_{j,i} - v_{i,j}, \tag{20.13}$$

where we used equation (A2.20). This tensor plays a prominent role in fluid kinematics, as we will now see.

21. *The Cauchy–Stokes Decomposition Theorem*

Let us now analyze in detail the instantaneous motion of a fluid. Consider the nature of the flow field in the neighborhood of some point O, which is moving with velocity $\mathbf{v}^0$, and let $\boldsymbol{\xi}$ be a small displacement away from O. Then to first order

$$v_i(\boldsymbol{\xi}) = v_i^0 + v_{i,j}\xi^j, \tag{21.1}$$

which shows that the relative velocity can be expressed in terms of the *velocity gradient tensor* $v_{i,j}$. Decomposing this tensor into an antisymmetric and symmetric part (cf. §A2.5), we have

$$v_i(\xi) = v_i^0 + \tfrac{1}{2}(v_{i,j} - v_{j,i})\xi^j + \tfrac{1}{2}(v_{i,j} + v_{j,i})\xi^j. \tag{21.2}$$

Each of the terms on the right-hand side of (21.2) admits of a direct physical interpretation. First, we see that the element undergoes a *translation* at a velocity $\mathbf{v}^0$. To interpret the second term, consider first the rotation of a rigid body with an angular rate ω_R around some axis $\mathbf{n}$ through O; write $\boldsymbol{\omega}_R = \omega_R \mathbf{n}$. Then the linear velocity of any point at a position $\boldsymbol{\xi}$ in the body is

$$\mathbf{v}_R = \boldsymbol{\omega}_R \times \boldsymbol{\xi}. \tag{21.3}$$

Now the second term of (21.2) is a sum of the form $\tilde{\Omega}_{ji}\xi^i$ where $\tilde{\Omega}_{ji}$ is the antisymmetric tensor

$$\tilde{\Omega}_{ji} \equiv \tfrac{1}{2}(v_{i,j} - v_{j,i}). \tag{21.4}$$

As is shown in equation (A2.63), this particular sum can be rewritten as $\tilde{\boldsymbol{\omega}} \times \boldsymbol{\xi}$, where $\tilde{\boldsymbol{\omega}}$ is the vector dual of the tensor $\tilde{\boldsymbol{\Omega}}$. From (21.3) we then see that the physical interpretation of the second term of (21.2) is that in the vicinity of O the fluid *rotates* with an effective angular velocity $\tilde{\boldsymbol{\omega}}$. To calculate the rotation rate we use (21.4) and equation (A2.60) to find

$$\tilde{\omega}^i = \tfrac{1}{2}e^{ijk}\tilde{\Omega}_{jk} = \tfrac{1}{2}e^{ijk}v_{k,j} = \tfrac{1}{2}(\text{curl}\,\mathbf{v})^i = \tfrac{1}{2}\omega^i, \tag{21.5}$$

which shows that the local angular velocity $\tilde{\boldsymbol{\omega}}$ is equal to one half the vorticity of the flow.

The symmetric tensor

$$E_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}) \tag{21.6}$$

appearing in the third term of (21.2) is known as the *rate of strain* tensor, for, as we shall now see, it describes how a fluid element in the neighborhood of O is *deformed* by the flow. In Cartesian coordinates

$$\mathsf{E} = \begin{pmatrix} (\partial v_x/\partial x) & \tfrac{1}{2}[(\partial v_x/\partial y) + (\partial v_y/\partial x)] & \tfrac{1}{2}[(\partial v_x/\partial z) + (\partial v_z/\partial x)] \\ \tfrac{1}{2}[(\partial v_y/\partial x) + (\partial v_x/\partial y)] & (\partial v_y/\partial y) & \tfrac{1}{2}[(\partial v_y/\partial z) + (\partial v_z/\partial y)] \\ \tfrac{1}{2}[(\partial v_z/\partial x) + (\partial v_x/\partial z)] & \tfrac{1}{2}[(\partial v_z/\partial y) + (\partial v_y/\partial z)] & (\partial v_z/\partial z) \end{pmatrix} \tag{21.7}$$

The diagonal elements $E_{(i)(i)}$ are called the *normal rates of strain*. As shown clearly in Figure 21.1a for the velocity field $\mathbf{v} = (x, 0, 0)$, these components represent the rates of *stretching* or *contraction* along the (x, y, z) axes. The off-diagonal terms E_{ij} equal half the *rate of shear deformation*, that is, half the rate of decrease of the angles between the edges of a parallelepiped of fluid originally having its edges along the orthogonal coordinate axes. For example, Figure 21.2 shows that the rate of change of the angle α between two line elements dx and dy originally along the x and y axes is $(d\alpha/dt) = -[(\partial v_x/\partial y) + (\partial v_y/\partial x)]$.

From fundamental matrix theory we know that any real symmetric matrix can be *diagonalized* by a suitable rotation of axes, and that when this is done the diagonal elements of the transformed matrix are equal to the (real) *eigenvalues* of the original matrix. The eigenvalues λ of E are the roots of the *secular equation* (or *characteristic equation*) obtained by setting to zero the determinant

$$|\mathsf{E} - \lambda \mathsf{I}| = |E_{ij} - \lambda\,\delta_{ij}| = 0. \tag{21.8}$$

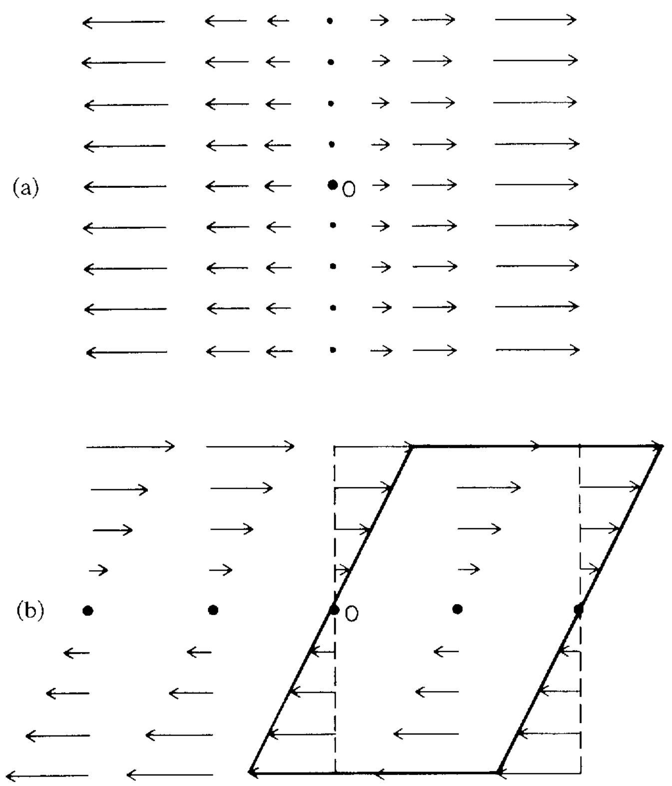

Fig. 21.1 (a) Stretching flow. (b) Flow with shear and vorticity.

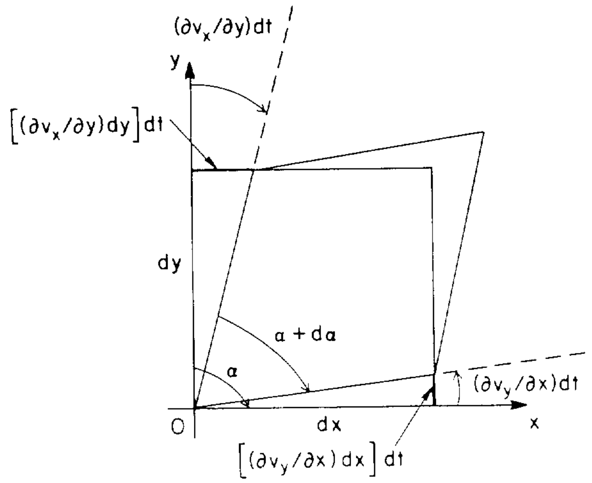

Fig. 21.2 Angular deformation of fluid element in shearing flow.

By expanding the determinant we find the cubic equation

$$\lambda^3 - A\lambda^2 + B\lambda - C = 0. \tag{21.9}$$

where A, B, and C are the three *invariants* (under rotation)

$$A = E_{11} + E_{22} + E_{33} = E_{ii} = \nabla \cdot \mathbf{v}, \tag{21.10}$$

$$B = E_{11}E_{22} + E_{22}E_{33} + E_{33}E_{11} - E_{12}^2 - E_{23}^2 - E_{31}^2 = \tfrac{1}{2}e_{abc}e_{ajk}E_{bj}E_{ck}, \tag{21.11}$$

and

$$C = |E_{ij}| = e_{ijk}E_{i1}E_{j2}E_{k3}. \tag{21.12}$$

The invariant A is clearly the *trace* of E; in this particular case it is equal to the divergence of the velocity field and is called the *dilatation* of the flow. Equation (21.9) yields three eigenvalues.

To show that A, B, and C are, in fact, invariant under rotations of coordinates we can proceed in two different ways. First, we can argue that because E is a definite physical entity regardless of the coordinate system chosen, the matrix E_{ij} must have unique eigenvalues; this will be true only if the secular equation is unique and hence its coefficients A, B, and C, are unique. Alternatively we can directly perform a transformation between two coordinate systems as described in §A2.4 and §A2.8. We find

$$\bar{A} = \bar{E}_{ii} = l_{pi}l_{qi}E_{pq} = \delta_{pq}E_{pq} = E_{pp} \equiv A, \tag{21.13}$$

$$\begin{aligned}\bar{B} &= \tfrac{1}{2}\bar{e}_{ijk}\bar{e}_{ipq}\bar{E}_{jp}\bar{E}_{kq} = \tfrac{1}{2}l_{ai}l_{bj}l_{ck}e_{abc}l_{di}l_{ep}l_{fq}e_{def}l_{sj}l_{tp}E_{st}l_{uk}l_{vq}E_{uv} \\ &= \tfrac{1}{2}\,\delta_{ad}\,\delta_{bs}\,\delta_{cu}\,\delta_{et}\,\delta_{fv}e_{abc}e_{def}E_{st}E_{uv} = \tfrac{1}{2}e_{abc}e_{aef}E_{be}E_{cf} \equiv B,\end{aligned} \tag{21.14}$$

and

$$\bar{C} = |\overline{E_{ij}}| = |l_{pi}l_{qj}E_{pq}| = |l_{pi}|\,|l_{qj}|\,|E_{pq}| = |E_{pq}| \equiv C, \tag{21.15}$$

where we have used equations (A2.12), (A2.29), and (A2.30).

Associated with each of the three eigenvalues of E is an eigenvector. If the eigenvalues are distinct, the eigenvectors are orthogonal; if not, the three eigenvectors can be orthogonalized. These three orthogonal vectors define the *principal axes* (or *principal directions*) relative to which the rate of strain tensor becomes diagonal. The eigenvalues themselves give the *principal rates of strain*, that is, the rates of stretching or contraction along the principal axes. Because the principal rates of strain are, in general, unequal, a spherical element will be deformed into an ellipsoid, as we will now show.

Consider the expression $f = \frac{1}{2}(v_{k,j}x^j x^k)$ where $\mathbf{x}$ is an infinitesimal displacement from O; this expression is quadratic in the coordinates x^i, and hence in general represents an ellipsoid. The vector normal to the surface $f =$ constant is given by ∇f. Consider a region so small that $v_{k,j}$ can be taken to be constant; then the components of ∇f are

$$\tfrac{1}{2}(v_{k,j}x^j x^k)_{,i} = \tfrac{1}{2}v_{k,j}(\delta_i^j x^k + \delta_i^k x^j) = \tfrac{1}{2}(v_{i,j} + v_{j,i})x^j = E_{ij}x^j. \tag{21.16}$$

But this expression is just the contribution from the rate of strain tensor to the velocity of **x** relative to O. Thus we conclude that the velocity associated with strain of the fluid is normal to the level surfaces

$$v_{k,j}x^jx^k = \text{constant} \tag{21.17}$$

which define the *rate of strain ellipsoid*. Along the principal axes, which are normal to the surface of this ellipsoid, the fluid moves in the same direction as the axes, and therefore the principal axes remain mutually perpendicular during the deformation. Thus in these directions, and only in these directions, the fluid undergoes a pure expansion or contraction.

We can summarize our analysis of the motion of a fluid by the *Cauchy-Stokes decomposition theorem*: at each point in the flow the instantaneous state of motion of a fluid can be resolved into a translation, plus a dilatation along three mutually perpendicular axes, plus a rigid rotation of these axes.

2.2 Equations of Motion and Energy

22. The Stress Tensor

The forces acting on a fluid may be divided into two types. First, *body forces*, arising from external agents such as gravitation, act throughout the whole volume of an element of fluid. Second, *surface forces* act on a volume element of fluid at its boundary surface; we describe these surface forces in terms of fluid *stresses* acting on the surface. As we will see in Chapter 3, fluid stress results from the transport of momentum within the fluid by molecular motions; an example is the pressure within a gas.

Consider a planar material surface (i.e., composed of a definite set of particles) dS located at position **x** in the fluid, and oriented with normal **n**. Let **t** be the surface force across this element exerted by the fluid on one side (the side containing **n**) on the fluid on the other side. In general, this force will depend not only on the surface element's position **x**, but also on its orientation **n**, so we write the total force that the surface element experiences as $\mathbf{t}(\mathbf{x}, \mathbf{n})\, dS$. Furthermore, a moment's reflection shows that the force exerted across the surface element by the fluid on the side away from **n** on the fluid on the other side is $-\mathbf{t}(\mathbf{x}, \mathbf{n})\, dS$. But we could also regard this force as $\mathbf{t}(\mathbf{x}, -\mathbf{n})\, dS$; hence we conclude that $\mathbf{t}(\mathbf{x}, -\mathbf{n}) = -\mathbf{t}(\mathbf{x}, \mathbf{n})$, that is, that **t** is an antisymmetric function of **n**.

Because the directions of **t** and **n** do not generally coincide, we infer that the fluid stresses that give rise to **t** must have associated with them two independent sets of directional information, hence we conjecture that stress must be a second-rank tensor. To demonstrate that this is so, suppose that an elementary tetrahedron located at a point P finds itself in equilibrium under the action of surface forces in the fluid. (For the moment, the requirement of equilibrium is arbitrary, but in §23 we shall see that the stresses at a point in a fluid are, in fact, always in equilibrium.)

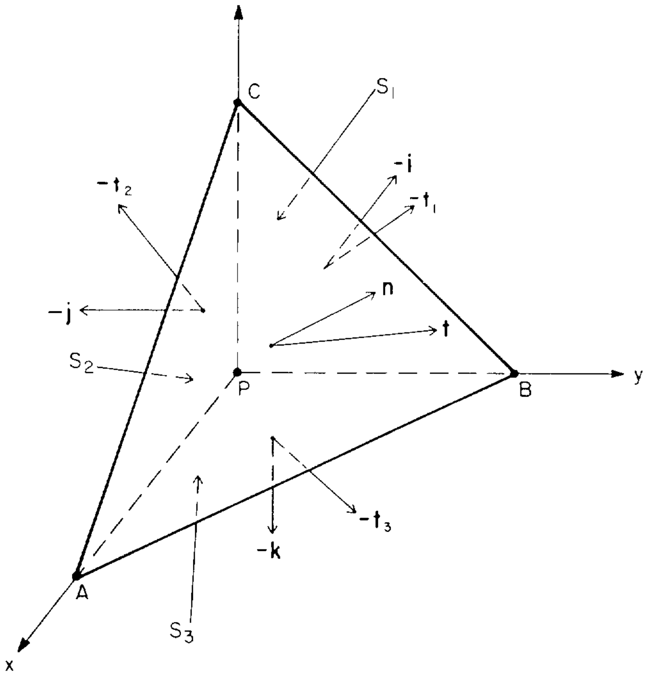

Fig. 22.1 Stresses acting on elementary tetrahedron.

As shown in Figure 22.1, choose three faces of the tetrahedron to lie in the coordinate planes, and let the slant face ABC with normal $\mathbf{n}$ have surface area S. The surface areas of the other three faces are evidently $S_1 = n_1 S$, $S_2 = n_2 S$, and $S_3 = n_3 S$, where n_l is the lth component of $\mathbf{n}$. Next, as above, let the surface force acting on a face with normal $\mathbf{m}$ be $\mathbf{t}(\mathbf{m})$. In particular, for notational convenience, write $\mathbf{t}(\mathbf{i}) \equiv \boldsymbol{t}_1$, $\mathbf{t}(\mathbf{j}) \equiv \boldsymbol{t}_2$, and $\mathbf{t}(\mathbf{k}) \equiv \boldsymbol{t}_3$. Then, noticing that S_1, S_2, and S_3 are oriented along $-\mathbf{i}$, $-\mathbf{j}$, and $-\mathbf{k}$, respectively, we see that to achieve equilibrium we must have

$$\mathbf{t}(\mathbf{n})S + \mathbf{t}(-\mathbf{i})S_1 + \mathbf{t}(-\mathbf{j})S_2 + \mathbf{t}(-\mathbf{k})S_3 \equiv 0, \tag{22.1a}$$

or, using the fact that $\mathbf{t}(\mathbf{n})$ is an antisymmetric function of $\mathbf{n}$,

$$\mathbf{t}(\mathbf{n})S - \boldsymbol{t}_1 S_1 - \boldsymbol{t}_2 S_2 - \boldsymbol{t}_3 S_3 \equiv 0. \tag{22.1b}$$

Expressing S_i in terms of n_i and S we then have

$$\mathbf{t}(\mathbf{n}) = n_1 \boldsymbol{t}_1 + n_2 \boldsymbol{t}_2 + n_3 \boldsymbol{t}_3. \tag{22.2}$$

Now let t_i denote the ith component of $\mathbf{t}(\mathbf{n})$, and let T_{ji} denote the ith component of $\boldsymbol{t}_j$. Then (22.2) states that

$$t_i = n^i T_{ji}. \tag{22.3}$$

Because $\mathbf{t}$ and $\mathbf{n}$ are independent vectors, we conclude that T_{ji} must be the components of a second-rank tensor T called the *stress tensor.* The diagonal elements of T are called *normal stresses,* which is appropriate because $T_{(i)(i)}$ gives the normal component of the surface force acting on a surface element that is oriented perpendicular to the ith coordinate axis. The off-diagonal elements are called *tangential stresses* (or *shearing stresses*). To be consistent with the theory of elasticity, it is conventional to take stress to be positive when it exerts a tension force and negative when it exerts a compressive force on a body.

For a fluid at rest we know from experiment that every element of area experiences only a force normal to the surface of the element, and that this force is independent of the orientation of the element. Such an isotropic normal stress with zero tangential stress is called a *hydrostatic stress.* The surface force **t** in this case must be proportional to **n**, with the constant of proportionality independent of the orientation of **n**; because we know empirically that the force in a fluid is always compressive (i.e., fluids do not support tension forces), we write the constant as $-p$. Then

$$t_j = T_{ij}n^i = -pn_j, \tag{22.4}$$

from which we recognize that p is to be identified with the hydrostatic pressure. The only form for T that can guarantee (22.4) for arbitrary **n** is

$$T_{ij} = -p\,\delta_{ij} \tag{22.5}$$

or

$$\mathsf{T} = -p\mathsf{I} \tag{22.6}$$

which is indeed an isotropic stress. Equation (22.6) applies to all fluids *at rest.*

If the fluid is *ideal,* then it is nonviscous and will not support tangential stress even when the fluid is in motion. The stress acting across a surface in an ideal fluid is thus always normal to the surface; hence (22.5) or (22.6) gives the stress tensor in an ideal fluid whether it is in motion or not. A *nonideal fluid* (i.e., a *real* fluid) will not support tangential stress when it is at rest, but can do so when it is in motion; for such fluids the stress tensor takes a more general form:

$$\mathsf{T} = -p\mathsf{I} + \boldsymbol{\sigma} \tag{22.7}$$

or

$$T_{ij} = -p\,\delta_{ij} + \sigma_{ij}. \tag{22.8}$$

The tensor $\boldsymbol{\sigma}$ is called the *viscous stress tensor*; we discuss it in some detail in §§25, 30, and 32.

23. *The Momentum Equation*

CAUCHY'S EQUATION OF MOTION

Consider a material volume $\mathcal{V}$ whose boundary surface lies entirely within the fluid. Then the principle of *conservation of linear momentum* asserts that the time rate of change of the momentum associated with the material element equals the total force acting on it; that is,

$$\frac{D}{Dt}\int_{\mathcal{V}} \rho \mathbf{v}\, dV = \int_{\mathcal{V}} \mathbf{f}\, dV + \int_{S} \mathbf{t}\, dS. \tag{23.1}$$

On the right-hand side of (23.1) the first term accounts for body forces and the second for surface forces. Now using the identity (19.12) we have

$$\int_{\mathcal{V}} \rho(Dv^i/Dt)\, dV = \int_{\mathcal{V}} f^i\, dV + \int_{S} t^i\, dS, \tag{23.2}$$

and thus from (22.3) and the divergence theorem (A2.67),

$$\int_{\mathcal{V}} \rho(Dv^i/Dt)\, dV = \int_{\mathcal{V}} f^i\, dV + \int_{S} T^{ij} n_j\, dS = \int_{\mathcal{V}} (f^i + T^{ij}_{\ ,j})\, dV. \tag{23.3}$$

Because the volume element $\mathcal{V}$ is arbitrary, equality of these integrals can be guaranteed only if their integrands are identical. We thus obtain *Cauchy's equation of motion*

$$\rho(Dv^i/Dt) = f^i + T^{ij}_{\ ,j} \tag{23.4}$$

or

$$\rho \mathbf{a} = \rho(D\mathbf{v}/Dt) = \mathbf{f} + \nabla \cdot \mathsf{T}. \tag{23.5}$$

In deriving (23.5) we have made no special assumptions about the physical mechanisms producing the stresses, and therefore the equation is quite general.

For an ideal fluid, (23.5) reduces to

$$\rho(D\mathbf{v}/Dt) = \mathbf{f} - \nabla p \tag{23.6}$$

or

$$\rho(Dv_i/Dt) = \rho(v_{i,t} + v^j v_{i,j}) = f_i - p_{,i}, \tag{23.7}$$

which is known as *Euler's equation of motion* or *Euler's momentum equation*. It is worth noticing here that the familiar pressure gradient term on the right-hand side of (23.6) or (23.7) is not, strictly speaking, the gradient of a scalar, but is in reality the divergence of an isotropic diagonal stress tensor.

SYMMETRY PROPERTIES OF THE STRESS TENSOR

Armed with (23.2) we are now in a position to prove that the stresses in a fluid are in equilibrium. Suppose a volume element has a characteristic dimension l; then its volume V scales as l^3, while its surface area S scales as l^2. Thus we see that

$$l^{-2}\int_{S} \mathbf{t}\, dS = \mathrm{O}(l), \tag{23.8}$$

and hence in the limit as $l \longrightarrow 0$ the integral vanishes. As this integral equals the sum of the surface forces over the surface, our assertion is proved.

Furthermore, by invoking the principle of *conservation of angular momentum*, we can prove that *the stress tensor is symmetric*. Thus, demanding that the time rate of change of the angular momentum of a material element equal the total applied torque we find

$$\frac{D}{Dt}\int_{\mathcal{V}} \rho(\mathbf{x}\times\mathbf{v})\,dV = \int_{\mathcal{V}} \rho\left(\mathbf{x}\times\frac{D\mathbf{v}}{Dt}\right) dV = \int_{\mathcal{V}} (\mathbf{x}\times\mathbf{f})\,dV + \int_{S} (\mathbf{x}\times\mathbf{t})\,dS, \tag{23.9}$$

where we used (19.12) and the fact that $(\mathbf{v}\times\mathbf{v})\equiv 0$. In (23.9), $\mathbf{x}$ is the position vector from the fixed origin of coordinates. Now writing (23.9) in component form, and applying the divergence theorem (A2.67) to the surface integral, we find

$$\begin{aligned}\int_{\mathcal{V}} e_{ijk}x^j[\rho(Dv^k/Dt)-f^k]\,dV &= \int_S e_{ijk}x^j t^k\,dS = \int_S e_{ijk}x^j T^{lk} n_l\,dS \\ &= \int_{\mathcal{V}} (e_{ijk}x^j T^{lk})_{,l}\,dV.\end{aligned} \tag{23.10}$$

Because $\mathcal{V}$ is arbitrary, (23.10) implies that the integrands must be equal, hence

$$e_{ijk}x^j[\rho(Dv^k/Dt)-f^k-T^{lk}_{\ ,l}]-e_{ijk}x^j_{\ ,l}T^{lk}=0. \tag{23.11}$$

But the first term is identically zero by virtue of Cauchy's equation. Hence

$$e_{ijk}x^j_{\ ,l}T^{lk} = e_{ijk}\,\delta^j_l T^{lk} = e_{ijk}T^{jk} = 0. \tag{23.12}$$

Writing out the last equality in (23.12) in components we have $(T_{12}-T_{21})=0$, $(T_{23}-T_{32})=0$, $(T_{31}-T_{13})=0$, whence it is obvious that

$$T_{ij} = T_{ji}, \tag{23.13}$$

which proves that T is symmetric.

The validity of the result just obtained hinges on the tacit assumption that the fluid is incapable of transporting angular momentum (and hence stress couples) across a surface by some internal process on the microscopic level; such fluids are called *nonpolar*, and include most gases. Certain non-Newtonian fluids (cf. §25) and polyatomic gases *can* transport angular momentum on the microscopic scale; these are called *polar* fluids. Such fluids have both a symmetric and an antisymmetric part to their stress tensors [see e.g., (**A1**, §5.11 and §5.13) for further discussion of these points]. Henceforth in this book we will deal exclusively with nonpolar fluids.

THE MOMENTUM FLUX DENSITY

The momentum equation can be cast into another extremely useful form. Taking the sum of the equation of motion

$$\rho v^i_{\ ,t} = f^i - \rho v^j v^i_{\ ,j} + T^{ij}_{\ ,j} \tag{23.14}$$

and the product of the velocity with the equation of continuity

$$v^i\rho_{,t} = -v^i(\rho v^j)_{,j} \tag{23.15}$$

we find

$$(\rho v^i)_{,t} = f^i - (\rho v^i v^j)_{,j} + T^{ij}_{,j} = f^i - \Pi^{ij}_{,j} \tag{23.16}$$

where

$$\Pi^{ij} \equiv \rho v^i v^j - T^{ij}. \tag{23.17}$$

When integrated over a fixed volume in space, equation (23.16) gives

$$\frac{\partial}{\partial t}\int_V \rho v^i\, dV = \int_V f^i\, dV - \int_V \Pi^{ij}_{,j}\, dV = \int_V f^i\, dV - \int_S \Pi^{ij} n_j\, dS. \tag{23.18}$$

The left-hand side is the rate of change of the *i*th component of momentum contained in the fixed volume. The first term on the right-hand side accounts for the rate of momentum transfer by external forces, and the second must be interpreted as the rate of momentum flow out through the bounding surface (obvious for $f^i = 0$).

Thus Π_{ij} gives the rate of flow of the *i*th component of the momentum through a unit area oriented normal to the *j*th coordinate axis; it is therefore called the *momentum flux-density tensor.* In particular, for an ideal fluid

$$\Pi_{ij} = \rho v_i v_j + p\,\delta_{ij}, \tag{23.19}$$

so that

$$\Pi_{ij} n^j = \rho v_i v_j n^j + p n_i = [\rho \mathbf{v}(\mathbf{v}\cdot\mathbf{n}) + p\mathbf{n}]_i \tag{23.20}$$

which shows that Π_{ij} accounts for momentum transport in the fluid by both macroscopic flow and microscopic particle motions. We will see in Chapter 3 that the same results can be derived from kinetic theory and also can be extended to include viscous terms. In Chapter 4 we will see that Π_{ij} has a natural relativistic generalization.

CURVILINEAR COORDINATES

The covariant generalization of Cauchy's equation of motion is

$$\rho a^i = f^i + T^{ij}_{;j}. \tag{23.21}$$

The covariant generalization of Euler's equation of motion is simply

$$\rho a_i = f_i - p_{,i}. \tag{23.22}$$

We cannot proceed further with Cauchy's equation until we specify in detail the form of the stress tensor; we therefore defer discussion of (23.21) until §26. In Euler's equation we need only use the correct expressions for the acceleration (derived in §15) and for ∇p. For example, in spherical coordinates we have

$$\rho\left[\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{1}{r}(v_\theta^2+v_\phi^2)\right]=f_r-\frac{\partial p}{\partial r}, \quad (23.23\text{a})$$

$$\rho\left[\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{1}{r}(v_rv_\theta-v_\phi^2\cot\theta)\right]=f_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}, \quad (23.23\text{b})$$

$$\rho\left[\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_\phi}{r}(v_r+v_\theta\cot\theta)\right]=f_\phi-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}. \quad (23.23\text{c})$$

For one-dimensional, spherically symmetric flow under a gravitational force $f_r=-G\mathcal{M}\rho/r^2$, we have simply

$$(\partial v_r/\partial t)+v_r(\partial v_r/\partial r)=-\rho^{-1}(\partial p/\partial r)-G\mathcal{M}/r^2. \quad (23.24)$$

For steady flow, the first term on the left-hand side is identically zero, and $(\partial/\partial r)$ can be replaced by (d/dr); we can then integrate (23.24) explicitly, [see (23.37) below].

HYDROSTATIC EQUILIBRIUM

In a *static* medium $\mathbf{v}=\mathbf{a}\equiv 0$, hence Euler's equation reduces to

$$\nabla p=\mathbf{f}, \quad (23.25)$$

from which we can determine the pressure stratification in the fluid. For instance, consider a plane-parallel atmosphere with homogeneous layers parallel to the (x, y) plane, stratified under a force $\mathbf{f}=-\rho g\mathbf{k}$ where g is the (constant) acceleration of gravity. Then (23.25) implies that $p=p(z)$ and

$$(dp/dz)=-\rho g. \quad (23.26)$$

Suppose the gas has a mean molecular weight μ and obeys the perfect gas law. Then in an *isothermal atmosphere* we can write

$$(dp/dz)=-(g\mu m_{\text{H}}/kT)p\equiv -p/H, \quad (23.27)$$

where H is the *pressure scale height.* In the atmosphere of the Sun H is of the order of 100 km; for the atmosphere of an early-type star it is a few thousand kilometers. Integrating (23.27) we have

$$p=p_0\exp[-(z-z_0)/H] \quad (23.28\text{a})$$

or

$$\rho=\rho_0\exp[-(z-z_0)/H]. \quad (23.28\text{b})$$

Equation (23.28) is valid only for an isothermal atmosphere, for which the scale height is constant; but it can be generalized to an atmosphere with a temperature gradient by writing

$$p = p_0 \exp\left\{-\int_{z_0}^{z} [g\mu m_H/kT(z)]\,dz\right\} = p_0 \exp\left[-\int_{z_0}^{z} dz/H(z)\right]. \tag{23.29}$$

In this case the density is most easily computed from the perfect gas law, given $p(z)$ and $T(z)$.

Notice that if we use the *column-mass* $dm \equiv -\rho\,dz$ as the independent variable, equation (23.26) becomes

$$(dp/dm) = g \tag{23.30}$$

whence

$$p = gm + p_0. \tag{23.31}$$

Here m is measured downward into the atmosphere (i.e., in the direction of gravity). Equation (23.31) holds whether or not the material is isothermal and shows why the column mass is the natural choice of coordinate for problems of hydrostatic equilibrium.

BERNOULLI'S EQUATION

From the vector identity (A2.59) with $\mathbf{a} = \mathbf{b} = \mathbf{v}$ we find

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \tfrac{1}{2}\nabla v^2 - \mathbf{v}\times(\nabla\times\mathbf{v}), \tag{23.32}$$

hence

$$(D\mathbf{v}/Dt) = (\partial\mathbf{v}/\partial t) - \mathbf{v}\times(\nabla\times\mathbf{v}) + \nabla(\tfrac{1}{2}v^2), \tag{23.33}$$

which is known as *Lagrange's acceleration formula.* Suppose now that the external force $\mathbf{f}$ acting on the fluid is conservative so that it can be written as the gradient of a potential Φ, that is, $\mathbf{f} = -\rho\,\nabla\Phi$. Then Euler's equation of motion for an ideal fluid becomes

$$(\partial\mathbf{v}/\partial t) - \mathbf{v}\times(\nabla\times\mathbf{v}) + \nabla(\tfrac{1}{2}v^2) + \rho^{-1}\,\nabla p + \nabla\Phi = 0. \tag{23.34}$$

We can derive an important integral of this equation by integrating *along a streamline.* Let $d\mathbf{s}$ be an element of length along any path. Then for any quantity α,

$$d\mathbf{s}\cdot\nabla\alpha = \alpha_{,i}\,dx^i \equiv d\alpha \tag{23.35}$$

where $d\alpha$ is the change in α along $d\mathbf{s}$. Furthermore, if $d\mathbf{s}$ lies along a streamline, it is parallel to $\mathbf{v}$, hence $d\mathbf{s}\cdot[\mathbf{v}\times(\nabla\times\mathbf{v})]$ must be identically zero. Using these results in (23.24) we see that in a conservative field of force the integral of Euler's equation along a streamline is

$$\int (\partial\mathbf{v}/\partial t)\cdot d\mathbf{s} + \tfrac{1}{2}v^2 + \int dp/\rho + \Phi = \text{constant} = C(t). \tag{23.36}$$

This equation is valid for unsteady flow, but applies only instant by instant because, in general, a given streamline in unsteady flows is composed of different sets of fluid particles at different times.

For *steady flow* (23.36) simplifies to

$$\tfrac{1}{2}v^2+\int dp/\rho+\Phi=C \tag{23.37}$$

where now C is a constant for all time along each streamline, but may differ from streamline to streamline. For *incompressible flow* (ρ = constant) we obtain *Bernoulli's equation*

$$\tfrac{1}{2}v^2+(p/\rho)+\Phi=C. \tag{23.38}$$

Explicit forms of (23.37) can also be obtained for *barotropic flow*, in which ρ is a function of p only. For example, for a polytropic gas $p \propto \rho^\gamma$ and we find

$$\tfrac{1}{2}v^2+[\gamma/(\gamma-1)](p/\rho)+\Phi=C. \tag{23.39}$$

For an isothermal flow $p \propto \rho$ and (23.37) yields

$$\tfrac{1}{2}v^2+(kT/\mu m_{\rm H})\ln\rho+\Phi=C \tag{23.40a}$$

or

$$\tfrac{1}{2}v^2+(kT/\mu m_{\rm H})\ln p+\Phi=C. \tag{23.40b}$$

Another general integral of the momentum equation can be written for the case of *irrotational flow*, for which $\mathbf{\nabla}\times\mathbf{v}=0$ and therefore $\mathbf{v}=\mathbf{\nabla}\phi$. Then, for barotropic flow (23.34) becomes

$$\mathbf{\nabla}\left[(\partial\phi/\partial t)+\tfrac{1}{2}v^2+\int dp/\rho+\Phi\right]=0. \tag{23.41}$$

We can now integrate (23.41) along an *arbitrary flow line* to obtain

$$(\partial\phi/\partial t)+\tfrac{1}{2}v^2+\int dp/\rho+\Phi=C(t), \tag{23.42}$$

where $C(t)$ is a function of time, but has the same value everywhere in the flow field. Equation (23.42) is known as the *Bernoulli-Euler equation* for potential flow. It is worth emphasizing that (23.37) holds along a streamline for steady flow, while (23.42) holds along an arbitrary line for a time-dependent but irrotational flow.

KELVIN'S CIRCULATION THEOREM

In §20 we derived Kelvin's equation for the time rate of change of the circulation

$$(D\Gamma/Dt)=\oint_C (D\mathbf{v}/Dt)\cdot d\mathbf{x}, \tag{23.43}$$

where C is a closed material curve. Substituting for $(D\mathbf{v}/Dt)$ from Euler's equation in a conservative force field we have

$$(D\Gamma/Dt)=-\oint_C (\mathbf{\nabla}\Phi)\cdot d\mathbf{x}-\oint_C \rho^{-1}(\mathbf{\nabla}p)\cdot d\mathbf{x}=-\oint_C d\Phi-\oint_C dp/\rho. \tag{23.44}$$

The first integral on the right-hand side is always zero because the potential is a single-valued point function. The second integral will be zero if the fluid is incompressible or is barotropic so that $(dp/\rho)=df(p)$. From (23.44) we thus obtain *Kelvin's circulation theorem* (or the *law of conservation of circulation*), which states that the circulation around any closed material path in a barotropic (or isothermal) flow of a perfect fluid moving in a conservative force field is independent of time. Such flows are called *circulation preserving.*

A consequence of Kelvin's theorem is that if the flow around some material path is initially irrotational, it remains irrotational. For flows of viscous fluids, particularly in the presence of material boundary surfaces or bodies moving through the fluid, this statement is not generally true, and vortices can be formed in thin boundary layers and then shed into the fluid [see, e.g., (**O1,** 70–72), (**S1**), and (**Y1**, 342–344)].

24. *The Energy Equation*

In addition to conservation laws for the mass and momentum of a fluid element, we can formulate a law of *conservation of energy*, which says that the rate at which the energy of a material element increases equals the rate at which heat is delivered to that element minus the rate at which it does work against its surroundings. Clearly this is merely a restatement of the first law of thermodynamics. In an ideal fluid there are no microscopic processes of energy dissipation from internal friction (viscosity), or of energy transport from one set of particles to another (thermal conduction). Moreover, at present we are disregarding heat exchange with external sources or by other physical mechanisms (e.g., radiation), hence there is a total absence of heat transfer between different parts of the fluid. We therefore conclude that *the motion of an ideal fluid must necessarily be adiabatic.*

TOTAL ENERGY AND MECHANICAL ENERGY EQUATIONS

For a material volume $\mathcal{V}$, the mathematical formulation of the energy conservation principle stated above is

$$\frac{D}{Dt}\int_{\mathcal{V}} \rho(e+\tfrac{1}{2}v^2)\,dV=\int_{\mathcal{V}} \mathbf{f}\cdot\mathbf{v}\,dV+\int_{S} \mathbf{t}\cdot\mathbf{v}\,dS. \tag{24.1}$$

The term on the left-hand side of (24.1) is the rate of change of the internal plus kinetic energy of the material element, and the terms on the right-hand side are the rates at which work is done by external forces and fluid stresses respectively. From (22.4) and the divergence theorem the rightmost term becomes, for an ideal fluid,

$$\int_S \mathbf{t}\cdot\mathbf{v}\,dS=-\int_S p\mathbf{v}\cdot\mathbf{n}\,dS=-\int_{\mathcal{V}} \boldsymbol{\nabla}\cdot(p\mathbf{v})\,dV. \tag{24.2}$$

Then using (19.12) to transform the left-hand side of (24.1), we have

$$\int_{\mathcal{V}} \{\rho[D(e+\tfrac{1}{2}v^2)/Dt]+\nabla\cdot(p\mathbf{v})-\mathbf{f}\cdot\mathbf{v}\}\,dV=0; \tag{24.3}$$

because $\mathcal{V}$ is arbitrary this implies

$$\rho[D(e+\tfrac{1}{2}v^2)/Dt]+\nabla\cdot(p\mathbf{v})=\mathbf{f}\cdot\mathbf{v}. \tag{24.4}$$

This is the *total energy equation* for an ideal fluid.

Now applying (19.13) to (24.4) we find the alternative forms

$$\partial(\rho e+\tfrac{1}{2}\rho v^2)/\partial t+\nabla\cdot[(\rho e+p+\tfrac{1}{2}\rho v^2)\mathbf{v}]=\mathbf{f}\cdot\mathbf{v}, \tag{24.5}$$

or

$$\partial(\rho e+\tfrac{1}{2}\rho v^2)/\partial t+\nabla\cdot[(h+\tfrac{1}{2}v^2)\rho\mathbf{v}]=\mathbf{f}\cdot\mathbf{v}, \tag{24.6}$$

where h is the specific enthalpy. Equation (24.6) can be integrated over a fixed volume V to obtain

$$\frac{\partial}{\partial t}\int_V \rho(e+\tfrac{1}{2}v^2)\,dV=-\int_S (h+\tfrac{1}{2}v^2)\rho\mathbf{v}\cdot d\mathbf{S}+\int_V \mathbf{f}\cdot\mathbf{v}\,dV. \tag{24.7}$$

In physical terms, the left-hand side of (24.7) is the rate of change of the total fluid energy in the fixed volume. The right-hand side is the rate of work being done by external forces, minus a term that must be interpreted as the rate of energy flow through the boundary surface of the volume. Hence $(h+\frac{1}{2}v^2)\rho\mathbf{v}$ must be the *energy flux density vector*; we will obtain a relativistic generalization of this expression in Chapter 4.

By forming the dot product of $\mathbf{v}$ with (23.6), Euler's equation, we find

$$\rho\mathbf{v}\cdot(D\mathbf{v}/Dt)=\tfrac{1}{2}\rho[D(\mathbf{v}\cdot\mathbf{v})/Dt]=\rho[D(\tfrac{1}{2}v^2)/Dt]=\mathbf{v}\cdot\mathbf{f}-(\mathbf{v}\cdot\nabla)p \tag{24.8}$$

which is the *mechanical energy equation* for an ideal fluid. Physically, this equation states that the time rate of change of the kinetic energy of a fluid element equals the rate of work done on that element by applied forces (both external and pressure).

GAS ENERGY EQUATION; FIRST AND SECOND LAWS OF THERMODYNAMICS

Subtracting the mechanical energy equation (24.8) from the total energy equation (24.4) we obtain the *gas energy equation*

$$\rho(De/Dt)+p(\nabla\cdot\mathbf{v})=0, \tag{24.9}$$

or, using the continuity equation (19.3),

$$(De/Dt)+p[D(1/\rho)/Dt]=0. \tag{24.10}$$

An alternative form of (24.10) is

$$\rho(Dh/Dt)-(Dp/Dt)=0. \tag{24.11}$$

We immediately recognize that (24.10) is merely a restatement of the first law of thermodynamics

$$(De/Dt)+p[D(1/\rho)/Dt]=T(Ds/Dt) \tag{24.12}$$

in the case that

$$(Ds/Dt)=(\partial s/\partial t)+(\mathbf{v}\cdot\nabla)s\equiv 0. \tag{24.13}$$

This result is completely consistent with our expectation, stated earlier, that the flow of an ideal fluid must be adiabatic. Combining (24.13) with the equation of continuity we have an *equation of entropy conservation* for ideal fluid flow:

$$\partial(\rho s)/\partial t+\nabla\cdot(\rho s\mathbf{v})=0. \tag{24.14}$$

We can therefore interpret $\rho s\mathbf{v}$ as an *entropy flux density*.

If the entropy is initially constant throughout some volume of a perfect fluid then it remains constant for all times within that material volume during its subsequent motion, and (24.13) reduces to $s=\text{constant}$; such a flow is called *isentropic*. A flow in which the entropy is constant and is uniform throughout the entire region under consideration is called *homentropic*.

The second law of thermodynamics for a material volume implies that

$$\frac{D}{Dt}\int_{\mathscr{V}}\rho s\,dV\geq\int_{\mathscr{V}}\frac{\rho}{T}\left(\frac{Dq}{Dt}\right)dV \tag{24.15}$$

where (Dq/Dt) is the rate of heat exchange of the fluid element with its surroundings. For an ideal fluid having no heat exchange with external sources this implies

$$\frac{D}{Dt}\int_{\mathscr{V}}\rho s\,dV\geq 0. \tag{24.16}$$

From what has been said thus far we would conclude that the equality sign must hold in (24.16) for the flow of an ideal fluid. However as we will see in Chapter 5, if we allow discontinuities (e.g., shocks) in the flow, then there can be an increase in entropy across a discontinuity even in an ideal fluid. Even though the entropy increase is correctly predicted mathematically by the equations of fluid flow for such *weak solutions*, it can be interpreted consistently, from a physical point of view, only when we admit the possibility of dissipative processes in the fluid.

CURVILINEAR COORDINATES

The covariant generalization of (24.6) is

$$[\rho(e+\tfrac{1}{2}v^2)]_{,t}+[\rho(h+\tfrac{1}{2}v^2)v^i]_{;i}=f_iv^i. \tag{24.17}$$

For example, in spherical coordinates we have

$$\frac{\partial}{\partial t}[\rho(e+\tfrac{1}{2}v^2)]+\frac{1}{r^2}\frac{\partial}{\partial r}[r^2\rho v_r(h+\tfrac{1}{2}v^2)] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}[\rho v_\theta \sin\theta(h+\tfrac{1}{2}v^2)] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}[\rho v_\phi(h+\tfrac{1}{2}v^2)]=\mathbf{f}\cdot\mathbf{v}. \tag{24.18}$$

For a one-dimensional, spherically symmetric flow the terms in $(\partial/\partial\theta)$ and $(\partial/\partial\phi)$ vanish identically.

STEADY FLOW

For one-dimensional steady flow we can write an explicit integral of the energy equation. Consider, for example, planar geometry with a gravity force $\mathbf{f}=-\rho g\mathbf{k}$ where g is constant. Then (24.6) becomes

$$d[\rho v_z(h+\tfrac{1}{2}v_z^2)]/dz=-\rho v_z g \tag{24.19}$$

which, recalling from (19.6) that $\dot{m}=\rho v_z=\text{constant}$, integrates to

$$\dot{m}(h+\tfrac{1}{2}v_z^2+gz)=\text{constant}. \tag{24.20}$$

Similarly, in spherical geometry, if $\mathbf{f}=-(G\mathcal{M}\rho/r^2)\hat{\mathbf{r}}$ we have

$$d[r^2\rho v_r(h+\tfrac{1}{2}v_r^2)]/dr=-(G\mathcal{M}/r^2)(r^2\rho v_r). \tag{24.21}$$

which, in view of (19.10), integrates to

$$\dot{\mathcal{M}}[h+\tfrac{1}{2}v_r^2-(G\mathcal{M}/r)]=\text{constant}. \tag{24.22}$$

Both (24.20) and (24.22) have a simple physical meaning: in a steady flow of an ideal fluid the *energy flux*, which equals the mass flux times the total energy (enthalpy plus kinetic plus potential) per unit mass, is a constant. We generalize these results to include the effects of viscous dissipation and thermal conduction in Chapter 3.

MATHEMATICAL STRUCTURE OF THE EQUATIONS OF FLUID DYNAMICS

We have formulated a total of five partial differential equations governing the flow of an ideal fluid: the continuity equation, three components of the momentum equation, and the energy equation. These relate six dependent variables: ρ, p, e, and the components of $\mathbf{v}$. To close the system we require *constitutive relations* that specify the thermodynamic properties of the material. We know that in general any thermodynamic property can be expressed as a function of two state variables. Thus we might choose an equation of state $p=p(\rho, T)$ and an equation for the internal energy, $e=e(\rho, T)$ sometimes called the *caloric equation of state*. These relations may be those appropriate to a perfect monatomic gas of structureless point particles having only translational degrees of freedom, as described in §§1, 4, and 8, or may include the effects of ionization and internal excitation as described in §§12 and 14.

We then have a total of seven equations in seven unknowns. This system of equations can be solved for the spatial variation of all unknowns as a function of time once we are given a set of *initial conditions* that specify the state and motion of the fluid at a particular time, plus a set of *boundary conditions* where constraints are placed on the flow. There are many techniques for solving these nonlinear equations. Analytical methods can yield solutions for some simplified problems, for example, certain incompressible flows, or low-amplitude flows for which the equations can be linearized. But, in general, this approach is too restrictive, and recourse must be had to numerical methods. In the linear regime numerical methods provide flexibility and generality in treating, for example, the depth-variation of the physical properties of the medium in which the flow occurs. In the nonlinear regime we encounter essential new physical phenomena: among them the formation of *shocks*. Here one can employ the *method of characteristics* or solve *difference-equation* representations of the fluid equations. The latter approach is both powerful and convenient, and is easy to explain. We defer further discussion of these methods until Chapter 5 where we discuss the additional physics needed in the context of specific examples.

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3

Dynamics of Viscous and Heat-Conducting Fluids

It is known from experiment that in all real fluids there are internal processes that result in transport of momentum and energy from one fluid parcel to another on a microscopic level. The momentum transport mechanisms give rise to *internal frictional forces* (*viscous forces*) that enter directly into the equations of motion, and that also produce frictional energy *dissipation* in the flow. The energy transport mechanisms lead to energy *conduction* from one point in the flow to another.

In this chapter we derive equations of fluid flow that explicitly account for the processes described above. To begin, we adopt a continuum view, which permits us to derive the mathematical form of the equations from quite general reasoning, drawing on heuristic arguments, symmetry considerations, and empirical facts. We then reexamine the problem from a microscopic kinetic-theory view, from which we recover essentially the same set of equations, but now with a much clearer understanding of the underlying physics. This approach also allows us to evaluate explicitly (for a given molecular model) the transport coefficients that are introduced on empirical grounds in the macroscopic equations.

3.1 Equations of Motion and Energy: The Continuum View

25. The Stress Tensor for a Newtonian Fluid

In §22 we showed that internal forces in a fluid can be described in terms of a stress tensor T_{ij}, which we showed to be symmetric, but otherwise left unspecified. We now wish to derive an explicit expression for the stress tensor in terms of the physical properties of the fluid and its state of motion. We can deduce the form of T_{ij} from the following physical considerations. (1) We expect internal frictional forces to exist only when one element of fluid moves relative to another; hence viscous terms must depend on the space derivatives of the velocity field, $v_{i,j}$. (2) We demand that the stress tensor reduce to its hydrostatic form when the fluid is at rest or translates uniformly (in which case it is at rest for an observer moving with the translation velocity). We therefore write

$$T_{ij} = -p\,\delta_{ij} + \sigma_{ij} \qquad (25.1)$$

where σ_{ij} is the *viscous stress tensor*, which accounts for the internal frictional forces in the flow. (3) For small velocity gradients we expect viscous forces, hence σ_{ij}, to depend only linearly on space derivatives of the velocity. A fluid that obeys this restriction is called a *Newtonian fluid.* [A discussion of *Stokesian fluids,* for which σ_{ij} depends quadratically on the velocity gradients, may be found in (**A1,** §§5.21 and 5.22). Non-Newtonian fluids of a very general nature are discussed in (**A2**).] (4) We expect viscous forces to be zero within an element of fluid in rigid rotation (because there is no slippage then). On these grounds, we expect no contribution to σ_{ij} from the vorticity tensor $\Omega_{ji} = (v_{i,j} - v_{j,i})$. We can exclude such a contribution for mathematical reasons as well because we have already shown that T_{ij}, hence σ_{ij}, must be symmetric [cf. (23.13)]. We therefore expect σ_{ij} to contain terms of the form $E_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$, which, as we saw in §21, describe the rate of strain in the fluid.

The most general symmetrical tensor of rank two satisfying the above requirements is

$$\sigma_{ij} = \mu(v_{i,j} + v_{j,i}) + \lambda v^{k}_{,k}\,\delta_{ij} = 2\mu E_{ij} + \lambda(\boldsymbol{\nabla}\cdot\mathbf{v})\,\delta_{ij}. \tag{25.2}$$

If we assume that the fluid is *isotropic,* so that there are no preferred directions, then λ and μ must be scalars; μ is called the *coefficient of shear viscosity* or the *coefficient of dynamical viscosity,* and λ is called the *dilatational coefficient of viscosity,* or the *second coefficient of viscosity.* For the present we regard both λ and μ as purely macroscopic coefficients that can be determined from experiment.

It is convenient to cast (25.2) into the slightly different form

$$\sigma_{ij} = \mu(v_{i,j} + v_{j,i} - \tfrac{2}{3}v^{k}_{,k}\,\delta_{ij}) + \zeta v^{k}_{,k}\,\delta_{ij}, \tag{25.3}$$

where

$$\zeta \equiv \lambda + \tfrac{2}{3}\mu \tag{25.4}$$

is known as the *coefficient of bulk viscosity.* The expression in parentheses in (25.3) has the mathematical property of being *traceless,* that is, it sums to zero when we contract on i and j. It also has the property of vanishing identically for a fluid that dilates symmetrically, that is, such that $(\partial v_1/\partial x^{(1)}) = (\partial v_2/\partial x^{(2)}) = (\partial v_3/\partial x^{(3)})$, and $(\partial v_i/\partial x^{j}) = 0$ for $i \neq j$. One can argue on intuitive physical grounds (**E2,** 19) that no frictional forces should be present in this case because there is no slipping of one part of the fluid relative to another; this will actually be true if and only if the coefficient of bulk viscosity ζ is identically zero.

Using (25.1) and (25.2) we see that the mean of the principal stresses is

$$\tfrac{1}{3}T_{ii} = -p + \zeta v^{k}_{,k}. \tag{25.5}$$

For an incompressible fluid $v^{k}_{,k} \equiv 0$, hence $-\frac{1}{3}T_{ii}$ equals the hydrostatic pressure p. For a compressible fluid we must identify p with the thermodynamic pressure given by the equation of state in order to be consistent with the requirements of hydrostatic and thermodynamic equilibrium.

If we call the mean of the principal stresses $-\bar{p}$, then

$$\bar{p}-p=-\zeta(\nabla\cdot\mathbf{v})=\zeta(D\ln\rho/Dt), \tag{25.6}$$

which shows that unless $\zeta\equiv 0$, there is in general a discrepancy between the scalar $\bar{p}$, which measures the isotropic part of the internal forces influencing the flow dynamics of a *moving* gas, and the thermostatic pressure p for the same gas *at rest* under identical thermodynamic conditions (i.e., same composition, density, temperature, ionization, etc.). This discrepancy can be quite significant for a gas undergoing very rapid (perhaps explosive) expansion or compression.

From kinetic theory one finds that for a perfect monatomic gas ζ is identically zero (cf. §32), as was first shown by Maxwell. The assumption that $\zeta\equiv 0$ for fluids in general was advanced by Stokes and is referred to as the *Stokes hypothesis*; the relation $\lambda=-\frac{2}{3}\mu$ (which implies $\zeta\equiv 0$) is called the *Stokes relation*. Fluids for which $\zeta=0$ are called *Maxwellian fluids*. In much of the classical work on the dynamics of viscous fluids the Stokes hypothesis is invoked from the outset. More recent work has been directed towards identifying the origins of bulk viscosity and its significance for fluid flow [e.g., see (**H2,** 521 and 644), (**O1,** 540–541), (**T1**), (**T3**), (**V1,** §10.8), and (**Z1,** 469)]. One finds that the bulk viscosity is nonzero (and indeed is of the same order of magnitude as the shear viscosity) when the gas undergoes a *relaxation process* on a time scale comparable to, or slower than, a characteristic fluid-flow time. Examples of such processes are the exchange of energy between translational motions and vibrational and rotational motions in polyatomic molecules, or between translational energy and ionization energy in an ionizing gas. When such processes occur, internal equilibration can lag flow-induced changes in the state of the gas, and this lag may give rise to irreversible processes that can, for example, cause the absorption and attenuation of sound waves (**L1,** §78) and affect the thermodynamic structure of shock waves (**Z1,** Chapter 7). In the derivations that follow we will not assume that $\zeta=0$, although this simplification will usually be made in later work. It should be noted that for incompressible flow $\nabla\cdot\mathbf{v}\equiv 0$, and the question of the correct value for ζ becomes irrelevant.

The covariant generalization of (25.1) valid in curvilinear coordinates is

$$T_{ij}=-pg_{ij}+\sigma_{ij}, \tag{25.7}$$

where from (25.3)

$$\sigma_{ij}=\mu(v_{i;j}+v_{j;i})+(\zeta-\tfrac{2}{3}\mu)v^{k}_{\ ;k}g_{ij}=2\mu E_{ij}+(\zeta-\tfrac{2}{3}\mu)v^{k}_{\ ;k}g_{ij}. \tag{25.8}$$

Here g_{ij} is the metric tensor of the coordinate system. Consider, for example, spherical coordinates. In what follows we shall need expressions for the rate of strain tensor E_{ij}, so we compute these first and then assemble T_{ij}. Thus using equation (A3.75), and retaining only the nonzero

Christoffel symbols given by equation (A3.63), we obtain

$$E_{11} = (\partial v_1/\partial r), \tag{25.9a}$$

$$E_{12} = \tfrac{1}{2}[(\partial v_1/\partial \theta) + (\partial v_2/\partial r)] - (v_2/r), \tag{25.9b}$$

$$E_{13} = \tfrac{1}{2}[(\partial v_1/\partial \phi) + (\partial v_3/\partial r)] - (v_3/r), \tag{25.9c}$$

$$E_{22} = (\partial v_2/\partial \theta) + r v_1, \tag{25.9d}$$

$$E_{23} = \tfrac{1}{2}[(\partial v_2/\partial \phi) + (\partial v_3/\partial \theta)] - v_3 \cot \theta, \tag{25.9e}$$

$$E_{33} = (\partial v_3/\partial \phi) + v_1 r \sin^2 \theta + v_2 \sin \theta \cos \theta. \tag{25.9f}$$

Now converting to physical components via equations (A3.46) and the analogues of (A3.47) appropriate to covariant components we have

$$E_{rr} = \frac{\partial v_r}{\partial r}, \tag{25.10a}$$

$$E_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right), \tag{25.10b}$$

$$E_{r\phi} = \frac{1}{2}\left(\frac{1}{r \sin \theta}\frac{\partial v_r}{\partial \phi} + \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r}\right), \tag{25.10c}$$

$$E_{\theta\theta} = \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}, \tag{25.10d}$$

$$E_{\theta\phi} = \frac{1}{2}\left(\frac{1}{r \sin \theta}\frac{\partial v_\theta}{\partial \phi} + \frac{1}{r}\frac{\partial v_\phi}{\partial \theta} - \frac{v_\phi \cot \theta}{r}\right), \tag{25.10e}$$

$$E_{\phi\phi} = \frac{1}{r \sin \theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r}. \tag{25.10f}$$

Finally, assembling T_{ij} in (25.7) and (25.8) we obtain

$$T_{rr} = -p + 2\mu\left(\frac{\partial v_r}{\partial r}\right) + (\zeta - \tfrac{2}{3}\mu)\boldsymbol{\nabla}\cdot\mathbf{v}, \tag{25.11a}$$

$$T_{\theta\theta} = -p + 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + (\zeta - \tfrac{2}{3}\mu)\boldsymbol{\nabla}\cdot\mathbf{v}, \tag{25.11b}$$

$$T_{\phi\phi} = -p + 2\mu\left(\frac{1}{r \sin \theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r}\right) + (\zeta - \tfrac{2}{3}\mu)\boldsymbol{\nabla}\cdot\mathbf{v}, \tag{25.11c}$$

$$T_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right], \tag{25.11d}$$

$$T_{r\phi} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) + \frac{1}{r \sin \theta}\frac{\partial v_r}{\partial \phi}\right], \tag{25.11e}$$

$$T_{\theta\phi} = \mu\left[\frac{\sin \theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin \theta}\right) + \frac{1}{r \sin \theta}\frac{\partial v_\theta}{\partial \phi}\right], \tag{25.11f}$$

where $(\boldsymbol{\nabla}\cdot\mathbf{v})$ is given by equation (A3.88).

26. *The Navier-Stokes Equations*

CARTESIAN COORDINATES

Cauchy's equation of motion (23.4) or (23.5) is extremely general because it makes no particular assumptions about the form of the stress tensor. If we specialize Cauchy's equation to the case of a Newtonian fluid by using equations (25.1) and (25.3), we obtain the *Navier-Stokes equations*, which are the equations most commonly employed to describe the flow of a viscous fluid. For Cartesian coordinates, we find, by direct substitution,

$$\rho(Dv_i/Dt) = \rho(v_{i,t} + v_j v_{i,j}) = f_i - p_{,i} + [\mu(v_{i,j} + v_{j,i})]_{,j} + [(\zeta - \tfrac{2}{3}\mu)v_{k,k}]_{,i}. \tag{26.1}$$

(Notice that this equation is not in covariant form, and applies only in Cartesian coordinates. By introducing the metric tensor we could write it in a form that is covariant when ordinary derivatives are changed to covariant derivatives, but this would require too cumbersome a notation for our present purposes.)

There are several somewhat simplified forms of (26.1) that are of interest. For example, for one-dimensional flows in planar geometry, (26.1) reduces to

$$\rho \frac{Dv_z}{Dt} = f_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left[(\tfrac{4}{3}\mu + \zeta)\frac{\partial v_z}{\partial z}\right]. \tag{26.2}$$

We notice that for this class of flows the shear viscosity and the bulk viscosity have the same dynamical effect, and that their combined influence on the momentum equation can be accounted for by using an effective viscosity coefficient $\mu' = (\mu + \frac{3}{4}\zeta)$.

Viscous forces in the equations of motion for one-dimensional flows are often described in terms of a fictitious *viscous pressure*

$$Q \equiv -\tfrac{4}{3}\mu'(\partial v_z/\partial z) = \tfrac{4}{3}\mu'(D \ln \rho/Dt), \tag{26.3}$$

which can make a significant contribution to the dynamical pressure during a rapid compression of the gas. The momentum equation can then be written

$$\rho(Dv_z/Dt) = f_z - [\partial(p + Q)/\partial z]. \tag{26.4}$$

We will see in §27 that μ' and Q as defined here appear again in the same forms in the energy equation, so that they provide a completely consistent formalism for treating viscous processes in one-dimensional planar flow problems. Furthermore, we will find in §59 that the concept of viscous pressure plays an important role in developing a technique for stabilizing the solution of flow problems with shocks by means of an *artificial viscosity* (or *pseudoviscosity*).

Another simple form of (26.1) results for flows in which the thermodynamic properties of the fluid do not change much from point to point so that the coefficients of viscosity can be taken to be constant. Then, in

Cartesian coordinates, we can write

$$\begin{aligned}\sigma_{ij,j} &= \mu(v_{i,j}+v_{j,i}-\tfrac{2}{3}v_{k,k}\,\delta_{ij})_{,j}+\zeta v_{k,ki}=\mu v_{i,jj}+(\zeta+\tfrac{1}{3}\mu)v_{k,ki}\\ &=\mu\,\nabla^2 v_i+(\zeta+\tfrac{1}{3}\mu)(\boldsymbol{\nabla}\cdot\mathbf{v})_{,i}\,, \end{aligned}\tag{26.5}$$

from which we can see that the equations of motion simplify to

$$\rho(D\mathbf{v}/Dt)=\mathbf{f}-\boldsymbol{\nabla}p+\mu\,\nabla^2\mathbf{v}+(\zeta+\tfrac{1}{3}\mu)\,\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}).\tag{26.6}$$

If the fluid is incompressible (a good approximation for liquids, and for gases at velocities well below the sound speed and over distances small compared to a scale height), then $\boldsymbol{\nabla}\cdot\mathbf{v}\equiv 0$ and the equations of motions reduce to

$$(D\mathbf{v}/Dt)=(\mathbf{f}-\boldsymbol{\nabla}p)/\rho+\nu\,\nabla^2\mathbf{v}\tag{26.7}$$

where $\nu\equiv\mu/\rho$ is called the *kinematic viscosity coefficient.*

CURVILINEAR COORDINATES

As mentioned in §23, the covariant generalization of Cauchy's equation is

$$\rho a^i=f^i+T^{ij}_{;j},\tag{26.8}$$

where a^i is the ith contravariant component of the acceleration as computed from (15.9). To evaluate (26.8) in any particular coordinate system, one can proceed as follows. (1) Calculate T_{ij} from (25.7) and (25.8). (2) Raise indices to obtain T^{ij}. (3) Use equation (A3.89) to evaluate the divergence $T^{ij}_{;j}$. (4) Convert to physical components using the expressions given in §A3.7. The calculations are straightforward but usually lengthy.

For spherical coordinates we can shorten this process by using the expressions for the physical components of T given by equation (25.11) directly in equations (A3.91) for the divergence. For a Newtonian fluid one finds, with a bit of patience,

$$\begin{aligned}\rho a_r = f_r &-\frac{\partial p}{\partial r}+\frac{\partial}{\partial r}\left[2\mu\frac{\partial v_r}{\partial r}+(\zeta-\tfrac{2}{3}\mu)(\boldsymbol{\nabla}\cdot\mathbf{v})\right]+\frac{1}{r}\frac{\partial}{\partial\theta}\left\{\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]\right\}\\ &+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left\{\mu\left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi}+r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]\right\}\\ &+\frac{\mu}{r}\left[4r\frac{\partial}{\partial r}\left(\frac{v_r}{r}\right)-\frac{2}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)-\frac{2}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right.\\ &\left.+r\cot\theta\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{\cot\theta}{r}\frac{\partial v_r}{\partial\theta}\right],\end{aligned}\tag{26.9a}$$

$$\begin{aligned}\rho a_\theta = f_\theta &-\frac{1}{r}\frac{\partial p}{\partial\theta}+\frac{\partial}{\partial r}\left\{\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]\right\}\\ &+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{2\mu}{r}\left(\frac{\partial v_\theta}{\partial\theta}+v_r\right)+(\zeta-\tfrac{2}{3}\mu)(\boldsymbol{\nabla}\cdot\mathbf{v})\right]\\ &+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left\{\mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right]\right\}\\ &+\frac{\mu}{r}\left\{\frac{2\cot\theta}{r}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\frac{v_\theta}{\sin\theta}\right)-\frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right]+3r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{3}{r}\frac{\partial v_r}{\partial\theta}\right\}.\end{aligned}\tag{26.9b}$$

and

$$
\begin{aligned}
\rho a_\phi = f_\phi &- \frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi} + \frac{\partial}{\partial r}\biggl\{\mu\biggl[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\biggr]\biggr\} \\
&+ \frac{1}{r}\frac{\partial}{\partial\theta}\biggl\{\mu\biggl[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\biggr]\biggr\} \\
&+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{2\mu}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial\phi} + v_r + v_\theta\cot\theta\right) + (\zeta - \tfrac{2}{3}\mu)(\nabla\cdot\mathbf{v})\right] \\
&+ \frac{\mu}{r}\biggl\{\frac{3}{r\sin\theta}\frac{\partial v_r}{\partial\phi} + 3r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) \\
&+ 2\cot\theta\biggl[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\biggr]\biggr\},
\end{aligned}
\tag{26.9c}
$$

where (a_r, a_θ, a_ϕ) are given by (15.11) and $\nabla\cdot\mathbf{v}$ is given by (A3.88).

Equations (26.9) are obviously very complicated, and, although they can actually be solved numerically with large high-speed computers, it is helpful to have simplified versions to work with. For example, expressions for zero bulk viscosity and constant dynamical viscosity are given in (**Y1,** 132), and for incompressible flow with constant viscosity in (**A1,** 183), (**L1,** 52), and (**Y1,** 132). A more useful simplification for our work is to consider a one-dimensional, spherically symmetric flow of a fluid with zero bulk viscosity. We then have

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r}\right) = f_r - \frac{\partial p}{\partial r} + \frac{1}{r^3}\frac{\partial}{\partial r}\left[\tfrac{4}{3}\mu r^4\frac{\partial}{\partial r}\left(\frac{v_r}{r}\right)\right]. \tag{26.10}$$

As we will see in §27, in spherical geometry it is not possible to account completely for viscous effects by means of a scalar viscous pressure, as can be done for planar flows.

27. The Energy Equation

TOTAL, MECHANICAL, AND GAS ENERGY EQUATIONS

Conservation of total energy for a material volume $\mathcal{V}$ in a viscous fluid implies that

$$\frac{D}{Dt}\int_{\mathcal{V}} \rho(e + \tfrac{1}{2}v^2)\,dV = \int_{\mathcal{V}} \mathbf{f}\cdot\mathbf{v}\,dV + \int_S \mathbf{t}\cdot\mathbf{v}\,dS - \int_S \mathbf{q}\cdot d\mathbf{S}. \tag{27.1}$$

Here the left-hand side gives the time rate of change of the internal plus kinetic energy contained within $\mathcal{V}$, while the three integrals on the right-hand side can be interpreted as (1) the rate at which work is done on the fluid by external volume forces, (2) the rate at which work is done by surface forces arising from fluid stresses, and (3) the rate of energy loss out of the fluid element by means of a direct transport mechanism having an *energy flux* $\mathbf{q}$. The sign of the last integral is taken to be negative because when $\mathbf{q}$ is directed along the outward normal $\mathbf{n}$ of S, heat is lost from $\mathcal{V}$. For the present we assume that $\mathbf{q}$ results from *thermal conduction*; in Chapter 7 we include radiative effects.

We can transform (27.1) into a more useful form by writing

$$\mathbf{t} \cdot \mathbf{v} = v_i t^i = v_i T^{ij} n_j \tag{27.2}$$

and using the divergence theorem to convert the surface integrals to volume integrals while transforming the left-hand side by use of (19.12). We then have

$$\int_{\mathcal{V}} \{\rho[D(e+\tfrac{1}{2}v^2)/Dt] - v_i f^i - (v_i T^{ij})_{,j} + q^i_{,i}\}\, dV = 0. \tag{27.3}$$

Because $\mathcal{V}$ is arbitrary, the integrand must vanish if the integral is to vanish, hence

$$\rho[D(e+\tfrac{1}{2}v^2)/Dt] = v_i f^i + (v_i T^{ij} - q^j)_{,j} \tag{27.4}$$

and therefore

$$(\rho e + \tfrac{1}{2}\rho v^2)_{,t} + [\rho(e+\tfrac{1}{2}v^2)v^j - v_i T^{ij} + q^j]_{,j} = v_i f^i, \tag{27.5a}$$

or, equivalently,

$$(\rho e + \tfrac{1}{2}\rho v^2)_{,t} + [\rho(h+\tfrac{1}{2}v^2)v^j - v_i \sigma^{ij} + q^j]_{,j} = v_i f^i. \tag{27.5b}$$

Equation (27.5) is the generalization of the *total energy equation* (24.6) to the case of a viscous fluid, and has a completely analogous interpretation term by term.

By forming the dot product of the velocity with Cauchy's equation of motion (23.4) we obtain the *mechanical energy equation* for a viscous fluid:

$$\rho v_i (Dv^i/Dt) = \rho[D(\tfrac{1}{2}v^2)/Dt] = v_i f^i + v_i T^{ij}_{,j}. \tag{27.6}$$

This is the generalization of (24.8) to the case of a viscous fluid.

Subtracting (27.6) from (27.4) we obtain the *gas energy equation*

$$\rho(De/Dt) = v_{i,j} T^{ij} - q^j_{,j}, \tag{27.7}$$

which is the generalization of (24.9) to the case of a viscous fluid. Using (25.7) and (25.8) we have

$$\begin{aligned} v_{i,j} T^{ij} &= v_{i,j}[-p\,\delta^{ij} + 2\mu E^{ij} + (\zeta - \tfrac{2}{3}\mu) v^k_{,k}\,\delta^{ij}] \\ &= -p v^i_{,i} + 2\mu v_{i,j} E^{ij} + (\zeta - \tfrac{2}{3}\mu)(v^i_{,i})^2. \end{aligned} \tag{27.8}$$

Recalling the definition of E^{ij} from (21.6). and in particular that it is symmetric in i and j, we see that the second term on the right-hand side reduces to $2\mu E_{ij} E^{ij}$. Thus we can write

$$v_{i,j} T^{ij} = -p(\boldsymbol{\nabla} \cdot \mathbf{v}) + \Phi \tag{27.9}$$

where the *dissipation function*

$$\Phi \equiv 2\mu E_{ij} E^{ij} + (\zeta - \tfrac{2}{3}\mu)(\boldsymbol{\nabla} \cdot \mathbf{v})^2 \tag{27.10}$$

accounts for *viscous energy dissipation* in the gas. Hence the gas energy equation becomes

$$\rho\left[\frac{De}{Dt}+p\frac{D}{Dt}\left(\frac{1}{\rho}\right)\right]=\Phi-\nabla\cdot\mathbf{q}. \tag{27.11}$$

In Cartesian coordinates

$$\begin{aligned}\Phi&=2\mu\left[\left(\frac{\partial v_x}{\partial x}\right)^2+\left(\frac{\partial v_y}{\partial y}\right)^2+\left(\frac{\partial v_z}{\partial z}\right)^2\right]\\&\quad+\mu\left[\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)^2+\left(\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}\right)^2+\left(\frac{\partial v_z}{\partial x}+\frac{\partial v_x}{\partial z}\right)^2\right]+(\zeta-\tfrac{2}{3}\mu)(\nabla\cdot\mathbf{v})^2.\end{aligned} \tag{27.12}$$

For one-dimensional planar flow this expression simplifies to

$$\Phi=(\tfrac{4}{3}\mu+\zeta)(\partial v_z/\partial z)^2, \tag{27.13}$$

or, in terms of the viscous pressure Q defined in (26.3),

$$\Phi=-Q(\partial v_z/\partial z)=-Q(\nabla\cdot\mathbf{v})=-\rho Q[D(1/\rho)/Dt]. \tag{27.14}$$

Thus (27.11) can be rewritten as

$$\rho\left[\frac{De}{Dt}+(p+Q)\frac{D}{Dt}\left(\frac{1}{\rho}\right)\right]=-\frac{\partial q_z}{\partial z}=\frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right) \tag{27.15}$$

which substantiates the claim made in §26 that the effects of viscosity in a one-dimensional planar flow can be accounted for completely and consistently through use of a viscous pressure Q. In writing the second equality in (27.15) we have anticipated equation (27.18).

ENTROPY GENERATION

By virtue of the first law of thrmodynamics,

$$T(Ds/Dt)=(De/Dt)+p[D(1/\rho)/Dt], \tag{27.16}$$

equation (27.12) can be transformed into the *entropy generation equation*

$$T(Ds/Dt)=\rho^{-1}(\Phi-\nabla\cdot\mathbf{q}). \tag{27.17}$$

We will prove below that the dissipation function is always greater than (or equal to) zero; we thus see from (27.17) that viscous dissipation leads to an irreversible increase in the entropy of the gas.

We can strengthen this statement by appealing to *Fourier's law* of heat conduction, which, on the basis of experimental evidence, states that the heat flux in a substance is proportional to the temperature gradient in the material, and that heat flows from hotter to cooler regions. That is,

$$\mathbf{q}=-K\,\nabla T, \tag{27.18}$$

where K is the *coefficient of thermal conductivity*. Using (27.18) in (27.17), and integrating over a material volume $\mathcal{V}$, we have

$$\begin{aligned}\int_{\mathcal{V}} \rho \frac{Ds}{Dt}\,dV &= \int_{\mathcal{V}} \frac{1}{T}[\nabla \cdot (K\,\nabla T)+\Phi]\,dV \\ &= \int_{\mathcal{V}} \left[\nabla \cdot \left(\frac{K}{T}\nabla T\right)+\frac{K}{T^2}(\nabla T)^2+\frac{\Phi}{T}\right] dV \qquad (27.19) \\ &= \int_S \frac{K}{T}(\nabla T) \cdot d\mathbf{S}+\int_{\mathcal{V}} \left[\frac{K}{T^2}(\nabla T)^2+\frac{\Phi}{T}\right] dV.\end{aligned}$$

The first term on the right-hand side of (27.19) is just

$$-\int_S T^{-1}\mathbf{q} \cdot \mathbf{n}\,dS = \int_{\mathcal{V}} T^{-1}(Dq/Dt)\,dV, \qquad (27.20)$$

that is, the rate at which heat is delivered into $\mathcal{V}$ divided by the temperature at which the delivery is made; in writing (27.20) we have assumed that $\mathcal{V}$ is infinitesimal so that the variation of T over S and within $\mathcal{V}$ can be neglected. The second term on the right-hand side of (27.19) is manifestly positive. Thus (27.19) is consistent with the second law of thermodynamics (24.15), and shows that *both* viscous dissipation and heat conduction within the fluid lead to an irreversible entropy increase.

THE DISSIPATION FUNCTION

We must now prove that Φ as given by (27.10) is, in fact, positive. We may omit the term $\zeta(\nabla \cdot \mathbf{v})^2$ from further consideration because it is obviously positive. We need consider, therefore, only

$$\Psi \equiv 2E_{ij}E^{ij}-\tfrac{2}{3}(\nabla \cdot \mathbf{v})^2. \qquad (27.21)$$

It is easy to show by direct substitution that

$$E_{ij}E^{ij}=A^2-2B, \qquad (27.22)$$

where A and B are the invariants defined by (21.10) and (21.11). Recalling that $A=\nabla \cdot \mathbf{v}$, we have

$$\Psi=\tfrac{4}{3}A^2-4B. \qquad (27.23)$$

Now that Ψ has been expressed in terms of invariants we may evaluate it in any coordinate system. Obviously it is most convenient to align the coordinate system along the principal axes of the rate-of-strain ellipsoid, in which case

$$\begin{aligned}\Psi &= \tfrac{4}{3}(E_{11}+E_{22}+E_{33})^2-4(E_{11}E_{22}+E_{22}E_{33}+E_{33}E_{11}) \\ &= \tfrac{4}{3}(E_{11}^2+E_{22}^2+E_{33}^2-E_{11}E_{22}-E_{22}E_{33}-E_{33}E_{11}).\end{aligned} \qquad (27.24)$$

Choose the largest element $E_{(i)(i)}=E_{\text{max}}$ as a normalization. Then we can write

$$\Psi=\tfrac{4}{3}E_{\text{max}}^2(1+\alpha^2+\beta^2-\alpha-\beta-\alpha\beta) \qquad (27.25)$$

where $\alpha \leq 1$ and $\beta \leq 1$. If $\alpha = \beta$, then

$$\Psi = \tfrac{4}{3}E_{\max}^2(1-2\alpha+\alpha^2) = \tfrac{4}{3}E_{\max}^2(1-\alpha)^2 \geq 0. \tag{27.26}$$

If $\alpha \neq \beta$, choose labels so that $\alpha > \beta$. Then

$$\Psi = \tfrac{4}{3}E_{\max}^2[(1-\alpha)^2 + (\alpha-\beta)(1-\beta)] \geq 0. \tag{27.27}$$

Thus in all cases $\Psi \geq 0$, which implies that $\Phi \geq 0$, which was to be proved.

Finally, it is easy to show by expanding the square that an alternative expression for Φ is

$$\Phi = 2\mu(E^{ij} - \tfrac{1}{3}\delta^{ij}\,\nabla\cdot\mathbf{v})(E_{ij} - \tfrac{1}{3}\delta_{ij}\,\nabla\cdot\mathbf{v}) + \zeta(\nabla\cdot\mathbf{v})^2. \tag{27.28}$$

One can see by inspection from this equation that $\Phi \geq 0$.

CURVILINEAR COORDINATES

The gas energy equation (27.11) is already in covariant form because the terms on the left-hand side are scalars, the divergence of the energy flux can be written as $\nabla\cdot\mathbf{q} = q^k_{;k}$, and Φ as defined by (27.10) is manifestly invariant under coordinate transformation. It is straightforward to evaluate these expressions in any coordinate system. Thus, in spherical coordinates,

$$\begin{aligned} -\nabla\cdot\mathbf{q} = \nabla\cdot(K\,\nabla T) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2K\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(K\sin\theta\frac{\partial T}{\partial\theta}\right) \\ &\quad + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(K\frac{\partial T}{\partial\phi}\right). \end{aligned} \tag{27.29}$$

Next, using equations (25.9) or (25.10) to evaluate $E_{ij}E^{ij}$, we easily find

$$\begin{aligned} \Phi = 2\mu\Bigg\{&\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} + \frac{v_r}{r} + \frac{v_\theta\cot\theta}{r}\right)^2 \\ &+ \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]^2 + \frac{1}{2}\left[r\frac{\partial}{\partial\phi}\left(\frac{v_\phi}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi}\right]^2 \\ &+ \frac{1}{2}\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right]^2\Bigg\} \\ &+ (\zeta - \tfrac{2}{3}\mu)\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right]^2. \end{aligned} \tag{27.30}$$

For the interesting case of one-dimensional, spherically symmetric flow with zero bulk viscosity, (27.30) simplifies to

$$\Phi = \tfrac{4}{3}\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_r}{r}\right)\right]^2, \tag{27.31}$$

hence the gas energy equation becomes

$$\rho\left[\frac{De}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right)\right] = \frac{1}{r^2}\frac{\partial}{\partial r}\left(Kr^2\frac{\partial T}{\partial r}\right) + \tfrac{4}{3}\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_r}{r}\right)\right]^2. \tag{27.32}$$

From a comparison of (26.10) and (27.32) one easily sees that in spherical geometry it is not possible to choose a scalar viscous pressure Q such that the viscous force can be written $(\partial Q/\partial r)$ and the viscous dissipation as $Q[D(1/\rho)/Dt]$, as was possible in planar geometry. The reason is that viscous effects originate from a *tensor*, which is not, in general, isotropic even for one-dimensional flow. Thus in §59 we find that the artificial viscosity terms used to achieve numerical stability in computations of spherical flows with shocks are best derived from a tensor artificial viscosity.

The covariant generalization of the total energy equation is easily obtained from (27.5) by replacing the ordinary partial derivative in the divergence with the covariant derivative. In general, the resulting expression is lengthy, but it becomes relatively simple for one-dimensional flow:

$$\begin{aligned}\frac{\partial}{\partial t}[\rho(e+\tfrac{1}{2}v_r^2)]+\frac{1}{r^2}\frac{\partial}{\partial r}\Big\{&r^2\rho v_r(e+\tfrac{1}{2}v^2)\\ &+r^2v_r\Big[p-\tfrac{4}{3}\mu r\frac{\partial}{\partial r}\Big(\frac{v_r}{r}\Big)\Big]-r^2K\frac{\partial T}{\partial r}\Big\}=f_rv_r.\end{aligned} \tag{27.33}$$

In (27.33) we have taken $\zeta\equiv 0$.

STEADY FLOW

For one-dimensional steady flow we can again write an explicit integral of the energy equation that now accounts for the effects of viscous dissipation and thermal conduction. In this case (27.5), for planar geometry and a constant gravitational acceleration, g reduces to

$$\frac{d}{dz}\left[\rho v_z\left(h+\tfrac{1}{2}v_z^2-\tfrac{4}{3}\nu\frac{dv_z}{dz}\right)-K\frac{dT}{dz}\right]=-\rho v_z g, \tag{27.34}$$

where h is the specific enthalpy, ν is the kinematic viscosity, and we have set $\zeta=0$. Then recalling that $\dot{m}=\rho v_z=\text{constant}$, we can integrate (27.34) to obtain

$$\dot{m}[h+\tfrac{1}{2}v_z^2+gz-\tfrac{4}{3}\nu(dv_z/dz)]-K(dT/dz)=\text{constant}. \tag{27.35}$$

Similarly, for a one-dimensional, spherically symmetric steady flow under the action of an inverse-square gravitational force, one finds from (27.33),

$$\dot{\mathcal{M}}\{h+\tfrac{1}{2}v_r^2-(G\mathcal{M}/r)-\tfrac{4}{3}\nu r[d(v_r/r)/dr]\}-4\pi r^2K(dT/dr)=\text{constant}, \tag{27.36}$$

where $\dot{\mathcal{M}}=4\pi r^2\rho v_r$; again we have set $\zeta=0$.

28. *Similarity Parameters*

As is clear from the material presented above and in Chapter 2, the equations of fluid dynamics are, in general, rather complicated. It is therefore useful to have a simple means for judging both the relative importance of various phenomena that occur in a flow, and the flow's qualitative nature. This is most easily done in terms of a set of dimensionless numbers which provide a convenient characterization of the dominant physical processes in the flow. These numbers are called *similarity parameters* because flows whose physical properties are such that they produce, in combination, the same values of those numbers can be expected to be qualitatively similar, even though the value of any one quantity—say velocity or a characteristic length—may be substantially different from one flow to another.

THE REYNOLDS NUMBER

A very basic and important flow parameter is the *Reynolds number*

$$\mathrm{Re} \equiv vl/\nu, \tag{28.1}$$

where v is a characteristic velocity in the flow, l is a characteristic length, and ν is the kinematic viscosity. Rewriting the term on the right-hand side as $(\rho v)v/(\mu v/l)$ we see that the Reynolds number gives a measure of the ratio of inertial forces (momentum flux per unit area) to viscous forces (viscosity times velocity gradient) in the flow. Limiting flow types characterized by the Reynolds number are inviscid flow—which occurs at an infinitely large Reynolds number—and inertialess viscous flow (Stokes flow)—which occurs at a vanishingly small Reynolds number.

The Reynolds number also determines whether a flow is *laminar* (i.e., smooth and orderly) or *turbulent* (i.e., disorderly and randomly fluctuating). Experiments show that the transition from laminar to turbulent flow occurs when the Reynolds number exceeds some critical value (which depends on the presence or absence, and the nature, of boundary surfaces or of solid bodies immersed in the flow). When a flow becomes turbulent, its kinematic and dynamical properties change radically and are extremely complicated to describe. We shall not deal with turbulence here but refer the reader to several excellent books on the subject [e.g., (**B1**), (**H1**), (**T2**)].

Both experiment and theoretical analysis show that when the Reynolds number is large, the bulk of a flow, except near physical boundary surfaces, is essentially inviscid and nonconducting. Major modifications of the nature of the flow by viscosity and thermal conductivity are generally confined to thin *boundary layers*. The ratio of the thickness δ of the boundary layer to a typical flow length l is given by

$$\delta/l \sim \mathrm{Re}^{-1/2} \tag{28.2}$$

[see e.g., (**A1,** 129), (**O1,** 548)]. The flows with which we shall deal have no association with solid boundary surfaces, hence we shall have no occasion to deal with boundary layer theory [see (**S1**)].

THE PRANDTL NUMBER

Another important similarity parameter for fluid flow is the *Prandtl number*

$$\mathrm{Pr} \equiv c_p \mu / K \tag{28.3}$$

where c_p is the specific heat at constant pressure, μ is the shear viscosity, and K is the thermal conductivity. Rewriting the right-hand side as $(\mu/\rho)/(K/\rho c_p)$ we see that Pr is the ratio of the kinematic viscosity to the thermal diffusivity. If Λ is a characteristic thickness for the thermal boundary layer, then we expect $(\rho c_p T) \sim (KT/\Lambda^2)$. From (28.2) we know that the thickness δ of the velocity boundary layer is $\delta \sim (\nu/vl)^{1/2}$. Hence $\mathrm{Pr} \propto \delta^2/\Lambda^2$. For gases, the Prandtl number is found to be of order unity so that the thicknesses of the thermal and velocity boundary layers are comparable; in §§29 and 33 we derive from kinetic theory an expression for Pr in monatomic perfect gases. For other fluids (e.g., oil), the Prandtl number is very large, which implies that the thermal boundary layer is very thin and that viscous effects predominate over thermal conduction in the flow.

THE PECLET NUMBER

The *Peclet number*

$$\mathrm{Pe} \equiv \rho c_p v l / K \tag{28.4}$$

is related to the Reynolds and Prandtl numbers by

$$\mathrm{Pe} = \mathrm{Re} \cdot \mathrm{Pr}. \tag{28.5}$$

Rewriting the right-hand side of (28.4) as $(\rho c_p T v)/(KT/l)$ we see that the Peclet number is essentially the ratio of heat transported by the flow to heat transported by thermal conduction. In this sense the Peclet number is to heat transport what the Reynolds number is to momentum transport, and indeed in nondimensionalized forms of the Navier-Stokes equations the Peclet number appears explicitly in the heat transfer terms in the same way the Reynolds number appears in the momentum transfer terms.

THE NUSSELT NUMBER

The *Nusselt number* is defined as

$$\mathrm{Nu} \equiv \alpha l / K, \tag{28.6}$$

where α is the *heat transfer coefficient*

$$\alpha \equiv q / \Delta T, \tag{28.7}$$

where in turn q is the heat flux per unit area and time through some surface area in the fluid, and $(\Delta T/l)$ measures the temperature gradient near that surface. Thus the Nusselt number gives a measure of the ratio of the total heat flux (by all mechanisms including, for example, radiation) to the conductive heat flux through a surface.

THE MACH NUMBER

A final extremely important parameter for gas dynamics is the *Mach number*

$$\mathrm{M} \equiv v/a, \tag{28.8}$$

where v is a characteristic flow speed and a is the adiabatic speed of sound (cf. §48). Flows in which $\mathrm{M} < 1$ are *subsonic* and flows in which $\mathrm{M} > 1$ are *supersonic*. Flows that make a transition from subsonic to supersonic, or vice versa, are called *transonic*. Flows at small Mach numbers can be treated as being locally (i.e., on scales much smaller than a gravitation-induced scale height) incompressible, because density variations in such flows are usually small. In flows where M approaches or exceeds unity, the nonlinearity of the fluid-dynamical equations induces large density variations, and the gas must be treated as compressible from the outset.

3.2 Equations of Motion and Energy: The Kinetic Theory View

The continuum view of fluids used above is quite satisfactory for the derivation of the basic equations of fluid dynamics, provided that we accept on empirical grounds the existence of viscous forces and conductive energy transport. But for gases the need to invoke empirical results can be avoided completely by adopting a kinetic theory view, which allows one to derive the mathematical forms of the viscous and thermal conduction terms from first principles. Furthermore, this approach allows one to evaluate the transport coefficients (e.g., μ and K) by direct calculation.

In what follows we confine attention entirely to dilute gases, as defined in §6 and §7. We first show qualitatively how transport phenomena arise in a gas and give semiquantitative estimates of their effects. We then derive the equations of gas dynamics, and give a rigorous first-order solution for the particle distribution function.

29. The Mean Free Path and Transport Phenomena

We showed in §6 that the motion of rigid-sphere particles in a neutral gas can be considered to be a sequence of straight-line paths terminated abruptly by collisions from which particles emerge on new paths. Each of these flights between collisions is called a *free path* of a particle. Because of the random nature of the collision process not all free paths have equal lengths. They will, nevertheless, have a well-defined average value, the *mean free path.*

If a gas is homogeneous and in equilibrium, we know from the principle of detailed balancing that the net effect of exchanges of particles from one element of the gas to another is exactly zero. If, however, the gas is out of equilibrium because of inhomogeneities in temperature, density, or fluid velocity (i.e., the average particle velocity at each point), then the particles originating from one fluid element may have a higher energy or momentum than those returning from an adjacent fluid element. In this event, there will be a net transfer of energy or momentum from one fluid element to another, which is interpreted on the macroscopic level as a viscous force or as thermal conduction.

In this section we make a rough calculation of these *transport phenomena* in terms of the mean free path. This approach yields considerable physical insight and also gives numerical results of the right order of magnitude. We give a more rigorous discussion of these processes in §§32 and 33.

THE MEAN FREE PATH

Consider two distinct species of rigid-sphere particles, A and B, interacting through collisions. The *bimolecular collision frequency* C_{AB} is the average total rate, per unit volume of gas, of collisions of all kinds between these species. The average *collision time* for either species is the reciprocal of the collision frequency per particle of that species; thus

$$\tau_A = n_A / C_{AB} \tag{29.1}$$

where n_A is the number density of particles of species A. The mean free path for particles of species A is the average distance they can travel in a collision time with an average speed $\langle U_A \rangle$, that is,

$$\lambda_A = \langle U_A \rangle \tau_A = n_A \langle U_A \rangle / C_{AB}. \tag{29.2}$$

Our first task is to calculate C_{AB}. Let the particles have diameters d_A and d_B so that their interaction diameter is $d_{AB} = \frac{1}{2}(d_A + d_B)$. From (7.21) we know that the cross section for rigid spheres of this effective diameter is $\sigma = \frac{1}{4} d_{AB}^2$. From exactly the same argument that leads to (7.32) we can write the total collision frequency as

$$C_{AB} = \frac{1}{4} \iiiint f_A f_B g d_{AB}^2 \sin\chi \, d\chi \, d\varepsilon \, d^3u_A \, d^3u_B, \tag{29.3}$$

where the symbols have the same meaning as in §§6 and 7. The integrals over χ and ε are trivial, yielding [cf. (7.23)]

$$C_{AB} = \pi d_{AB}^2 \iint f_A f_B g \, d^3u_A \, d^3u_B. \tag{29.4}$$

To compute C_{AB} we obviously must know the distribution functions f_A and f_B; we can obtain a reasonable estimate by using the equilibrium distribution function f_0 given by (8.9) so that

$$C_{AB} = n_A n_B (m_A m_B)^{3/2} (2\pi kT)^{-3} \pi d_{AB}^2 \times \iint \exp[-(m_A u_A^2 + m_B u_B^2)/2kT] g \, d^3u_A \, d^3u_B. \tag{29.5}$$

From (7.6) we see that the argument in the exponential can be rewritten in terms of g, the relative speed, and G, the speed of the center of mass, of the two particles. Furthermore, we have shown earlier [cf. (7.33) and

(7.34)] that $d^3u_{\mathrm{A}}\,d^3u_{\mathrm{B}} = d^3g\,d^3G$. Thus (29.5) becomes

$$C_{\mathrm{AB}} = 2n_{\mathrm{A}}n_{\mathrm{B}}(m_{\mathrm{A}}m_{\mathrm{B}})^{3/2}(kT)^{-3}d_{\mathrm{AB}}^2$$
$$\times \int_0^\infty e^{-(M_{\mathrm{AB}}G^2/2kT)}G^2\,dG \int_0^\infty e^{-(\tilde{m}_{\mathrm{AB}}g^2/2kT)}g^3\,dg. \quad (29.6)$$

The integrals are given in standard tables, and we obtain, finally,

$$C_{\mathrm{AB}} = n_{\mathrm{A}}n_{\mathrm{B}}d_{\mathrm{AB}}^2(8\pi kT/\tilde{m}_{\mathrm{AB}})^{1/2}. \quad (29.7)$$

Equation (29.7) gives the total collision rate per unit volume for two *distinct* species of particles A and B. If the gas consists of a single species so that all the particles are identical, then this rate should be divided by 2 because the integration counts a given pair of particles twice, once regarding one of them as having a velocity $\mathbf{u}_{\mathrm{A}}$ and the other a velocity $\mathbf{u}_{\mathrm{B}}$, and again with the first now having a velocity $\mathbf{u}_{\mathrm{B}}$ and the second a velocity $\mathbf{u}_{\mathrm{A}}$, when in fact both of these cases give the same collision. On the other hand, for the purpose of counting mean free paths, we recognize that each collision between two identical particles terminates *two* free paths of particles in the gas; this just cancels the factor of two mentioned above. Thus the collision time for species A is

$$\tau_{\mathrm{A}} = (\tilde{m}_{\mathrm{AB}}/8\pi kT)^{1/2}/(n_{\mathrm{B}}d_{\mathrm{AB}}^2) \quad (29.8)$$

whether species B is identical to species A or not. In the former case $\tilde{m}_{\mathrm{AB}} = \frac{1}{2}m$, $n_{\mathrm{B}} = N$, and $d_{\mathrm{AB}} = d$, so that

$$\tau = (m/16\pi kT)^{1/2}(Nd^2). \quad (29.9)$$

Now using (29.9) in (29.2), along with (8.12) for $\langle U\rangle$, we find

$$\lambda = (2^{1/2}N\pi d^2)^{-1}. \quad (29.10)$$

From purely dimensional arguments, we would estimate that for a particle moving with a speed $\langle U\rangle$ through a field of N stationary particles per cm^3, the collision time is $\tau \sim 1/N\langle U\rangle\sigma$, and hence $\lambda \sim 1/N\sigma$ where σ is the interaction cross section πd^2. The additional factor of $\sqrt{2}$ appearing in (29.10) simply accounts for the correct average relative speeds of the test and field particles when the field particles are also allowed to move.

THE DISTRIBUTION OF FREE PATHS

Let us now calculate the *distribution of free paths*. Consider a large number N_0 of particles at some instant. Let $N(x)$ be the number of particles that have not yet suffered collisions after each particle in the group has traveled a distance x along its free path. Then if P_c is the *average collision probability* per unit length along the path, we must have

$$dN(x) = -N(x)P_c\,dx, \quad (29.11)$$

whence

$$N(x) = N_0e^{-P_cx}. \quad (29.12)$$

But, by definition, the mean free path is the average distance a particle can travel between collisions; therefore the average collision probability per unit length must be $P_c = 1/\lambda$. Thus the number of particles that are able to travel a distance x without suffering a collision is

$$N(x) = N_0 \exp(-x/\lambda). \tag{29.13}$$

THE KNUDSEN NUMBER

When the mean free path λ is very small compared to a characteristic macroscopic length l in the flow, the gas can be considered to be a continuum and treated by the equations of continuum hydrodynamics. In the opposite extreme where $\lambda \gg l$, we are in the regime of *rarefied gas dynamics*, and must treat the problem of *free-molecule flow* in which the gas molecules act as if they were a large set of noninteracting particles. To categorize flows into a scheme between the extreme cases just described we introduce another dimensionless similarity parameter, the *Knudsen number*

$$\mathrm{Kn} \equiv \lambda/l. \tag{29.14}$$

As we shall see in §32, for small Knudsen numbers we can derive the equations of gas dynamics from a first-order solution of the Boltzmann transport equation. This solution yields analytical expressions for the energy flux vector and the fluid stress tensor in terms of local properties and their gradients. The resulting conservation equations contain the same macroscopic terms as the equations for ideal fluids, plus additional terms describing momentum and energy transport by microscopic processes in the gas.

On the other hand, for large Knudsen numbers the local description mentioned above no longer suffices, because we must account for efficient exchange of particles over mean free paths that are of the same size as, or larger than, characteristic flow lengths. We must then employ a rather elaborate transport theory formulated in terms of integrations over finite (in some cases quite large) volumes of the gas.

The Knudsen number is small, and hence the continuum treatment is valid, for the material component of the radiating fluid in all astrophysical flows we consider in this book, even in such rarefied flows as stellar winds (cf. §62). For example, in the solar atmosphere we have a neutral atomic hydrogen density $N \sim 10^{15}\,\mathrm{cm}^{-3}$, and can assign a typical cross section of $\sigma \sim \pi a_0^2 \sim 10^{-16}\,\mathrm{cm}^2$ to these particles, from which we estimate $\lambda \sim 10\,\mathrm{cm}$; this may be compared to a pressure scale height H, which is of the order of $10^7\,\mathrm{cm}$. Hence $\mathrm{Kn} \sim 10^{-6}$, and a purely local, continuum treatment of the gas dynamics suffices (except for the effects of the interaction of radiation with the material). For the parameters just quoted and a temperature of the order of 10^4 K the collision time is $\tau \sim 10^{-5}$ sec. Thus, local nonuniformities in temperature, density, or velocity on scales of a few cm are substantially reduced by molecular exchange in times of the order of 10^{-5} sec.

TRANSPORT PROCESSES

Let us now evaluate the coefficients of viscosity and thermal conduction in terms of the mean free path. The basic assumption made here is that particles typically transport a momentum (or energy) characteristic of their point of origin through a mean free path, and then deposit it into the gas at the end of their flight. Consider a uniform gas in which the macroscopic flow velocity is $\mathbf{v} = (v_x, 0, 0)$, where $v_x = v_x(z)$. Then from (25.3) the viscous stress tensor is

$$\boldsymbol{\sigma} = \mu \begin{pmatrix} 0 & 0 & (\partial v_x/\partial z) \\ 0 & 0 & 0 \\ (\partial v_x/\partial z) & 0 & 0 \end{pmatrix}. \tag{29.15}$$

Hence from (22.3) the viscous force exerted on a unit area oriented parallel to the (x, y) plane, with normal $\mathbf{n} = (0, 0, 1)$, at a height $z = z_0$ is

$$\mathbf{t}_{\text{visc}} = [\mu(\partial v_x/\partial z)_0, 0, 0]. \tag{29.16}$$

This result is compatible with our qualitative expectation that each layer tends to drag the adjacent layer (lying below it, for the choice of $\mathbf{n}$ we have made) in the direction of its motion relative to that layer.

From the kinetic theory viewpoint, the viscous force $\mathbf{t}_{\text{visc}}$ must equal the net rate of transport in the z direction of momentum in the x direction across a unit area parallel to the (x, y) plane, per unit time. The number of particles crossing this plane from below in a unit time is

$$N_+ = \int_0^\infty \Phi(U_z) U_z \, dU_z \tag{29.17}$$

where Φ is the one-dimensional distribution of particle velocities along the z axis. We assume, for purposes of calculating a first approximation for N_+, that we can use the equilibrium distribution given by (8.15), despite the fact that the gas is not really in equilibrium. The integral in (29.17) is then elementary and one finds

$$N_+ = \tfrac{1}{4}N(8kT/\pi m)^{1/2} = \tfrac{1}{4}N\langle U\rangle. \tag{29.18}$$

The number N_- crossing in the downward direction equals N_+ by symmetry.

Now the average x velocity of particles crossing the unit plane at $z = z_0$ is not $v_x(z_0)$, but rather corresponds to the average x velocity in the layer whence these particles started their free paths. On the average we expect this layer to be located at $z = z_0 \pm \zeta_\mu \lambda$, where ζ_μ is an unknown numerical factor of order unity. Then to first order the net transport of x momentum across the unit area in a unit time from the positive to the negative side is

$$\begin{aligned} &\tfrac{1}{4}N\langle U\rangle\{m[v_x(z_0) + \zeta_\mu \lambda(\partial v_x/\partial z)_0] - m[v_x(z_0) - \zeta_\mu \lambda(\partial v_x/\partial z)_0]\} \\ &\quad = \tfrac{1}{2}Nm\langle U\rangle \zeta_\mu \lambda(\partial v_x/\partial z)_0, \end{aligned} \tag{29.19}$$

which must equal the macroscopic force given by (29.16). Therefore we identify

$$\mu = \tfrac{1}{2}\zeta_\mu Nm\lambda\langle U\rangle = \tfrac{1}{2}\zeta_\mu \rho\lambda\langle U\rangle = (\zeta_\mu/\pi d^2)(mkT/\pi)^{1/2}. \tag{29.20}$$

As we will see in §33, this result is in fortuitously good agreement with the exact result for rigid-sphere particles, for which one finds $\zeta_\mu = (1.016)\times(5\pi/16) = 0.9975$. Equation (29.20) shows explicitly the remarkable result, first noted by Maxwell, that for a dilute gas the coefficient of shear viscosity is independent of density (which Maxwell verified by experiment). For a rigid-sphere gas, μ varies as $T^{1/2}$; we will see in §33 that the temperature dependence of μ for an ionized gas is quite different.

By a similar analysis we can evaluate K, the coefficient of thermal conductivity. We now suppose the gas is at rest and is uniform except for a temperature gradient in the z direction [i.e., $T = T(z)$]. From the macroscopic viewpoint, the heat flux will be

$$q_z = -K(\partial T/\partial z)_0. \tag{29.21}$$

From the kinetic theory viewpoint, the heat flux equals the net rate of thermal energy transport in the $+z$ direction by individual particles. The thermal energy per particle is $\bar{e} = \tfrac{3}{2}kT$. We assume that, on the average, particles originate in layers at $z = z_0 \pm \zeta_K\lambda$, where again ζ_K is an unknown numerical factor of order unity. The net thermal energy transport is, therefore,

$$\begin{aligned}\tfrac{1}{4}N\langle U\rangle\tfrac{3}{2}k\{[T_0 - \zeta_K\lambda(\partial T/\partial z)_0] - [T_0 + \zeta_K\lambda(\partial T/\partial z)_0]\}\\ = -\tfrac{3}{4}kN\langle U\rangle\zeta_K\lambda(\partial T/\partial z)_0,\end{aligned} \tag{29.22}$$

whence we identify

$$K = \tfrac{3}{4}\zeta_K kN\lambda\langle U\rangle, \tag{29.23}$$

or, in view of (8.20),

$$K = \tfrac{1}{2}\zeta_K\rho\lambda\langle U\rangle c_v. \tag{29.24}$$

Thus, from (29.20), we have

$$K = (\zeta_K/\zeta_\mu)c_v\mu. \tag{29.25}$$

The results obtained in §33 show that for a rigid-sphere gas $\zeta_K = (1.025)\times\tfrac{5}{2}\times(5\pi/16) = 2.5160$, and hence $(\zeta_K/\zeta_\mu) = 2.5225$.

The Prandtl number for a rigid-sphere gas follows immediately from (29.25):

$$\Pr = (c_p\mu/K) = (\zeta_\mu/\zeta_K)(c_p/c_v) \approx \tfrac{2}{5}\gamma, \tag{29.26}$$

where we have used the values of ζ_μ and ζ_K just quoted. For a perfect monoatomic gas we should thus have $\Pr \approx \tfrac{2}{3}$, which is in good agreement with experiment for noble gases (**H2,** 16). Equation (29.25) with $(\zeta_K/\zeta_\mu) \approx 2.5$ remains valid for power-law interactions in general and for Coulomb interactions in particular.

Although we have been able to obtain physical insight and even semi-quantitative results from mean-free-path calculations, the approach is obviously only heuristic. The calculation can be refined by allowing for differences in the lengths of free paths of different particles, effects of previous collisions on particle speeds ("persistence of velocity"), and correlation between the energy of a particle and the distance over which it can penetrate to the test area [see for example (**J1,** Chaps. 11–13), (**J2,** Chaps. 6 and 7)]. However, even in its most refined forms this approach remains ad hoc, applies only to rigid-sphere particles, and yields results of unknown reliability (their accuracy being determinable only when the calculation can be done by more precise methods). The basic flaw in this approach is that it does not provide a means of calculating the actual velocity distribution function of the particles, nor of specifying how the real distribution departs from the equilibrium distribution as a result of spatial variations in the macroscopic physical properties of the gas. Reliable results are obtained only when a direct calculation of the nonequilibrium velocity-distribution function is made along the lines discussed in §§32 and 33.

30. Moments of the Boltzmann Equation

The equations of gas dynamics can be deduced directly by calculating moments of the Boltzmann equation for quantities that are conserved in collisions of the particles composing the gas. This approach provides an independent derivation of the equations obtained earlier from macroscopic arguments, and deepens our understanding of the physical meaning of the terms that appear in these equations.

THE CONSERVATION THEOREM

We form a *moment* of the Boltzmann equation

$$(\partial f/\partial t)+u^i(\partial f/\partial x^i)+a^i(\partial f/\partial u^i)=(Df/Dt)_{\text{coll}} \tag{30.1}$$

by multiplying through by any physical quantity $Q(\mathbf{x}, \mathbf{u})$, and then integrating over velocity. Thus

$$\int Q\cdot\left(\frac{\partial f}{\partial t}+u^i\frac{\partial f}{\partial x^i}+a^i\frac{\partial f}{\partial u^i}\right)d^3u=\int Q\cdot\left(\frac{Df}{Dt}\right)_{\text{coll}}d^3u=I(Q), \tag{30.2}$$

where $I(Q)$ is defined by (7.39). If Q is *conserved* during the collision $(\mathbf{u}, \mathbf{u}_1)\rightarrow(\mathbf{u}', \mathbf{u}_1')$, in the sense that it is a summational invariant of the form

$$Q(\mathbf{x}, \mathbf{u})+Q(\mathbf{x}, \mathbf{u}_1)=Q(\mathbf{x}, \mathbf{u}')+Q(\mathbf{x}, \mathbf{u}_1'), \tag{30.3}$$

then it follows immediately from (7.39) and (7.42) that $I(Q)\equiv 0$, and (30.2) reduces to a *conservation law*. Henceforth, we assume that Q is conserved.

We can cast the left-hand side of (30.2) into a more useful form. Define the average value $\langle A\rangle$ of any quantity A as

$$\langle A\rangle=\left(\int Af\,d^3u\right)\Big/\left(\int f\,d^3u\right)=N^{-1}\int Af\,d^3u, \tag{30.4}$$

where the second equality follows from (6.1); here N is the number of particles per cm^3. Then, bearing in mind that t, x^i and u^i are all independent variables we have

$$\int Q\frac{\partial f}{\partial t}\,d^3u=\frac{\partial}{\partial t}\int Qf\,d^3u-\int\frac{\partial Q}{\partial t}f\,d^3u=\frac{\partial}{\partial t}(N\langle Q\rangle)-N\left\langle\frac{\partial Q}{\partial t}\right\rangle, \tag{30.5}$$

$$\begin{aligned}\int Qu^i\frac{\partial f}{\partial x^i}\,d^3u&=\frac{\partial}{\partial x^i}\int Qu^if\,d^3u-\int u^i\frac{\partial Q}{\partial x^i}f\,d^3u\\&=\frac{\partial}{\partial x^i}(N\langle Qu^i\rangle)-N\left\langle u^i\frac{\partial Q}{\partial x^i}\right\rangle,\end{aligned} \tag{30.6}$$

and

$$\begin{aligned}\int Qa^i\frac{\partial f}{\partial u^i}\,d^3u&=\int\frac{\partial}{\partial u^i}(Qa^if)\,d^3u-\int\frac{\partial}{\partial u^i}(Qa^i)f\,d^3u\\&=\sum_i\iint_{-\infty}^{\infty}du_l\,du_k\,\Big|_{-\infty}^{\infty}(Qa^if)-N\left\langle\frac{\partial(Qa^i)}{\partial u^i}\right\rangle=-N\left\langle\frac{\partial(Qa^i)}{\partial u^i}\right\rangle.\end{aligned} \tag{30.7}$$

In (30.7) we have made use of the fact that as $u^i\rightarrow\pm\infty$, $f\rightarrow 0$ so strongly that (Qa^if) vanishes for all Q's of interest. We thus obtain the general *conservation theorem*, which states that

$$\frac{\partial}{\partial t}(N\langle Q\rangle)+\frac{\partial}{\partial x^i}(N\langle Qu^i\rangle)-N\left[\left\langle\frac{\partial Q}{\partial t}\right\rangle+\left\langle u^i\frac{\partial Q}{\partial x^i}\right\rangle+\left\langle\frac{\partial(Qa^i)}{\partial u^i}\right\rangle\right]=0, \tag{30.8}$$

for any conserved quantity Q. If we now restrict attention to velocity-independent external forces, and assume that Q is a function of $\mathbf{u}$, but not an explicit function of $\mathbf{x}$ or t, then (30.8) simplifies to

$$\frac{\partial}{\partial t}(N\langle Q\rangle)+\frac{\partial}{\partial x^i}(N\langle Qu^i\rangle)-Na^i\left\langle\frac{\partial Q}{\partial u^i}\right\rangle=0. \tag{30.9}$$

Consider now a gas of particles having no internal structure. For such a gas we can immediately write down five conserved quantities:

$$Q_1=m,\qquad Q_2=mu_1,\qquad Q_3=mu_2,\qquad Q_4=mu_3,\qquad \text{and}\qquad Q_5=\tfrac{1}{2}mu^2. \tag{30.10}$$

These are all obviously collisional invariants in view of (7.39) and (7.42), which show that, for purposes of calculating $I(Q)$, they are equivalent to the summational invariants (7.6) and the statement (7.7) that the total mass of the collision partners is conserved. We shall now show that if we use these five collisional invariants in (30.9), we recover the equation of continuity, the three equations of motion, and the energy equation for the gas.

THE EQUATION OF CONTINUITY

Choose $Q = m$; then (30.9) is a statement of mass conservation, which asserts that

$$(Nm)_{,t} + (Nm\langle u^j\rangle)_{,j} = 0. \tag{30.11}$$

But $Nm = \rho$, the gas density, and $\langle u^j\rangle = v^j$, the jth component of the macroscopic flow velocity. Thus we recognize that (30.11) is simply the continuity equation

$$(\partial\rho/\partial t) + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0. \tag{30.12}$$

The vector $Nm\mathbf{v} = \rho\mathbf{v}$ can be interpreted physically either as the *momentum density vector* or as the *mass flux vector*.

THE MOMENTUM EQUATIONS

Now take $Q = mu^i$, the ith component of particle momentum; then (30.9) is an expression of momentum conservation, stating that

$$(Nm\langle u^i\rangle)_{,t} + (Nm\langle u^i u^j\rangle)_{,j} - Nma^j\,\delta^i_j = 0. \tag{30.13}$$

But $Nm\langle u^i\rangle = \rho v^i$, and $Nma^i = \rho a^i = f^i$, the ith component of the external force per unit volume. Furthermore, using (6.4) and the fact that $\langle\mathbf{U}\rangle \equiv 0$, the tensor $Nm\langle u^i u^j\rangle$ can be rewritten

$$\begin{aligned} Nm\langle u^i u^j\rangle &= \rho\langle(v^i + U^i)(v^j + U^j)\rangle = \rho(v^i v^j + v^i\langle U^j\rangle + v^j\langle U^i\rangle + \langle U^i U^j\rangle) \\ &= \rho v^i v^j + \rho\langle U^i U^j\rangle. \end{aligned} \tag{30.14}$$

This tensor can be identified as the *momentum flux density tensor* because it gives the product of Nu^i, the number flux of particles crossing a unit area oriented perpendicular to the ith coordinate axis, times mu^j, the jth component of their momentum, averaged over all particles. Thus $Nm\langle u^i u^j\rangle = \Pi^{ij}$, and by comparing (30.14) with (23.17) we see that we can now identify the stress tensor T^{ij} as

$$T^{ij} \equiv -\rho\langle U^i U^j\rangle. \tag{30.15}$$

Equation (30.15) shows explicitly that fluid stress results from momentum exchange on a microscopic level within the gas. Collecting results, we see that (30.13) reduces to

$$(\rho v^i)_{,t} + (\rho v^i v^j - T^{ij})_{,j} = f^i, \tag{30.16}$$

which is just the ith component of Cauchy's equation of motion (23.16) as derived earlier from macroscopic considerations.

As is done in macroscopic fluid dynamics, we define the negative of the pressure in the gas to be the average of the normal stresses. Thus

$$p = -\tfrac{1}{3}T_{ii} = \tfrac{1}{3}\rho\langle U_x^2 + U_y^2 + U_z^2\rangle = \tfrac{1}{3}\rho\langle U^2\rangle. \tag{30.17}$$

We can also define a translational kinetic temperature for particles in the gas via (8.13):

$$\langle U^2\rangle = 3kT/m. \tag{30.18}$$

Therefore we must have

$$p = \rho kT/m = NkT; \tag{30.19}$$

that is, the pressure is given by the perfect gas law, as expected for a gas of point particles. If we again write $T_{ij} = -p\,\delta_{ij} + \sigma_{ij}$ where σ_{ij} is the viscous stress tensor, we easily see from (30.15) and (30.18) that

$$\sigma_{ij} = -\rho(\langle U_iU_j\rangle - \tfrac{1}{3}\langle U^2\rangle\,\delta_{ij}), \tag{30.20}$$

which gives an explicit expression from which the viscous stresses can be calculated. It is clear that the results we obtain must necessarily depend on the functional form of the distribution function, which enters directly into the computation of the required averages of the velocity components. Using this decomposition of T_{ij} into a hydrostatic pressure and viscous stress, and recalling the equation of continuity, we can rewrite (30.16) into the standard form

$$\rho(Dv_i/Dt) = \rho(v_{i,t} + v_jv_{i,j}) = -p_{,i} + \sigma_{ij,j} + f_i \tag{30.21}$$

(for Cartesian coordinates only), which differs from its macroscopic counterpart (26.1) only in that we have not yet specified an analytic form for σ_{ij}.

THE ENERGY EQUATION

To obtain a conservation law for translational energy we set $Q = \tfrac{1}{2}mu^2$ in (30.9), which then becomes

$$(\tfrac{1}{2}Nm\langle u^2\rangle)_{,t} + (\tfrac{1}{2}Nm\langle u^2u^j\rangle)_{,j} = Nma^iv_j. \tag{30.22}$$

Now $Nm = \rho$; $Nma^iv_j = \rho a^iv_j = f^iv_j$;

$$\begin{aligned}\langle u^2\rangle &= \langle u_iu^i\rangle = \langle(v_i + U_i)(v^i + U^i)\rangle\\ &= v_iv^i + 2v_i\langle U^i\rangle + \langle U_iU^i\rangle = v^2 + \langle U^2\rangle;\end{aligned} \tag{30.23}$$

and

$$\begin{aligned}\langle u^2u^j\rangle &= \langle(v_i + U_i)(v^i + U^i)(v^j + U^j)\rangle\\ &= v_iv^i(v^j + \langle U^j\rangle) + v^j\langle U_iU^i\rangle + 2v^jv_i\langle U^i\rangle + 2v_i\langle U^iU^j\rangle + \langle U_iU^iU^j\rangle\\ &= v^j(v^2 + \langle U^2\rangle) + 2v_i\langle U^iU^j\rangle + \langle U^2U^j\rangle.\end{aligned} \tag{30.24}$$

Therefore (30.22) can be written

$$[\rho(\tfrac{1}{2}\langle U^2\rangle + \tfrac{1}{2}v^2)]_{,t} + [\rho(\tfrac{1}{2}\langle U^2\rangle + \tfrac{1}{2}v^2)v^j + \rho v_i\langle U^iU^j\rangle + \rho\langle\tfrac{1}{2}U^2U^j\rangle]_{,j} = v_jf^i. \tag{30.25}$$

Each of the terms in (30.25) admits of a simple physical interpretation. First, the translational energy associated with the random motion of a particle is $\tilde{e}_{\text{trans}} = \frac{1}{2}m\langle U^2\rangle$, which yields a specific internal energy (per gram) of $e = \frac{1}{2}\langle U^2\rangle$ for the gas. Next, from (30.15) we recall that $\rho\langle U^iU^j\rangle = -T^{ij} = p\,\delta^{ij} - \sigma^{ij}$. Finally, the last term on the left-hand side is the energy per particle, $\frac{1}{2}mU^2$, times the flux of particles NU^j along the jth axis in a frame moving with the fluid, averaged over all particles; this is the energy flux in the gas resulting from microscopic motions, hence we can identify the heat flux vector $\mathbf{q}$ as

$$\mathbf{q} = \rho\langle \tfrac{1}{2}U^2\mathbf{U}\rangle. \tag{30.26}$$

We then see that (30.25) can be rewritten as

$$(\rho e + \tfrac{1}{2}\rho v^2)_{,t} + [\rho(h + \tfrac{1}{2}v^2)v^j - v_i\sigma^{ij} + q^j]_{,j} = v_i f^i, \tag{30.27}$$

which is identical to the macroscopic total energy equation (27.5), and has the same physical interpretation as (27.5) and (24.5). Equation (30.27) can also be reduced, using the dot product of (30.21) with $\mathbf{v}$, to a gas energy equation identical to the macroscopic equation (27.11).

The conservation equations derived in this section are exact (for the adopted model of the gas), but have no practical value until we can evaluate σ^{ij} and q^i. In the macroscopic theory, this is done by an appeal to empirical results. Using kinetic theory we can evaluate these terms from first principles (see §§32 and 33).

31. *Conservation Equations for Equilibrium Flow*

In all flows of interest in astrophysical applications, both the velocity of the flow and the thermodynamic properties of the gas vary from point to point; indeed, gradients in the material properties such as pressure are often responsible (along with gravity) for driving the flow. Such flows are obviously not in equilibrium in the strict sense that the material is homogeneous and gradients of all physical quantities are zero. Nevertheless, if particle mean free paths are very small compared to characteristic flow lengths, and if gradients are sufficiently small, then the gas can be considered to be in *local equilibrium*. In this limit we assume that the translational energy-exchange time is negligible in comparison to characteristic flow times, and that all particles have a single Maxwellian velocity distribution at the local value of the temperature and density.

If we assume that $f(\mathbf{x}, \mathbf{u}, t) \equiv f_0(\mathbf{U})$, as given by (8.9) with $T = T(\mathbf{x})$ and $N = N(\mathbf{x})$, then we can evaluate q_i and σ_{ij} immediately from (30.26) and (30.20). Thus, for the heat flux

$$(q_i)_0 = \tfrac{1}{2}\rho(m/2\pi kT)^{3/2}\int_{-\infty}^{\infty} dU_k \int_{-\infty}^{\infty} dU_j \int_{-\infty}^{\infty} U_iU^2 \exp[-(mU^2/2kT)]\, dU_i, \tag{31.1}$$

(i, j, and k distinct; no sum on i). Now U^2 is an even function of all velocity components, while U_i is an odd function; therefore the integral over dU_i vanishes identically. Hence, in local equilibrium, $\mathbf{q}_0 \equiv 0$. Similarly, for an element of the viscous stress tensor

$$(\sigma_{ij})_0 = -\rho\left(\frac{m}{2\pi kT}\right)^{3/2} \int_{-\infty}^{\infty} e^{-(mU_k^2/2kT)}\, dU_k \times \int_{-\infty}^{\infty} U_j e^{-(mU_j^2/2kT)}\, dU_j \int_{-\infty}^{\infty} U_i e^{-(mU_i^2/2kT)}\, dU_i \tag{31.2}$$

(no sum on i or j). Here, the integrals over both dU_i and dU_j vanish identically because U_i and U_j are odd functions and the exponential is even. Therefore, in local equilibrium $(\sigma_{ij})_0 \equiv 0$, and $(T_{ij})_0 = -p\,\delta_{ij}$.

Using these results in the general conservation equations derived in §30, we obtain the gas dynamical equations for equilibrium flow:

$$\rho_{,t} + (\rho v^i)_{,i} = 0, \tag{31.3}$$

$$\rho(Dv_i/Dt) = \rho(v_{i,t} + v^j v_{i,j}) = f_i - p_{,i} \tag{31.4}$$

and

$$(\rho e + \tfrac{1}{2}\rho v^2)_{,t} + [\rho(h + \tfrac{1}{2}v^2)v^j]_{,j} = v_i f^i, \tag{31.5}$$

or, combining (31.4) and (31.5),

$$\rho(De/Dt) + p[D(1/\rho)/Dt] = 0, \tag{31.6}$$

which are simply the hydrodynamical equations for an ideal fluid derived in Chapter 2. We thus see that these equations provide only a high-order idealization of the flow of a real gas, applying in the limit that gradients of physical properties in the flow field are vanishingly small, and that the velocity distribution departs negligibly from the local Maxwellian distribution. We further see that it is logically inconsistent to use the equilibrium distribution function f_0 to calculate the effects of transport phenomena and to evaluate transport coefficients as was done in §29, inasmuch as we have just seen that these phenomena are absent if the distribution function f actually equals f_0, and arise only *because* f differs from f_0.

32. The Chapman–Enskog Solution for Nonequilibrium Flow

To calculate the distribution function f, we must solve the Boltzmann equation. As we saw in §§8 and 9, it is possible to do this exactly for strict equilibrium, in which case we obtain the Maxwellian distribution f_0. The solution for nonequilibrium regimes is difficult even in simple physical situations, and in general we can obtain only approximate results. There are many ways in which approximate solutions of the Boltzmann equation can be constructed, but the most useful for our present purposes is the *Chapman–Enskog method*, which provides a self-consistent expansion scheme that yields a series of successive approximations for the deviations of the true distribution function f from the equilibrium distribution f_0.

It should be stressed from the outset that the Chapman–Enskog method does not provide a completely general solution of the Boltzmann equation, but rather only a particular class of solutions that depend on local values and local gradients of the temperature, density, and fluid velocity. The importance of this class of solutions is that a distribution function initially not of this type tends to become one of the class in a time of the order of a collision time (**C2,** §7.2).

The Chapman–Enskog theory is moderately complicated, so we confine our attention only to the lowest level of approximation that yields results of interest. Thus we evaluate only the first-order term in the expansion, which leads to the Navier-Stokes equations, and, in doing so, we solve only to lowest order certain integral equations that appear in the development of the theory. Furthermore, we confine attention to a single-species monatomic gas; other limitations of the method will be discussed at the end of the section. Very complete treatments of the Chapman–Enskog method in higher orders of approximation and for multicomponent gases can be found in (**B2,** Chaps. 7 and 9), (**C2,** Chaps. 7–10, 15, and 18), (**H2,** Chaps. 7 and 9), and (**V1,** Chap. 10).

EXPANSION PROCEDURE

To economize the notation, let us write

$$\mathscr{D}f \equiv (\partial f/\partial t) + u^i(\partial f/\partial x^i) + a^i(\partial f/\partial u^i) \tag{32.1}$$

for the differential operator that appears in Boltzmann's equation, and

$$J(f_i, f_j) \equiv \int\int (f_i'f_j' - f_if_j)g\sigma(\mathbf{\Omega})\,d\Omega\,d^3u_j \tag{32.2}$$

for the collision integral of any two functions f_i and f_j.

To see how to develop an expansion procedure for solving Boltzmann's equation it is useful to transform the equation to a nondimensional form. Thus, express velocities in units of a reference speed v_0, lengths in units of a characteristic length l, time in units of (l/v_0), accelerations in units of (v_0^2/l), and densities in terms of a reference density n_0. Furthermore, express the differential collision rate, $ng\sigma\,d\Omega$, in units of a characteristic collision frequency ν_0. One then finds

$$\xi\hat{\mathscr{D}}\hat{f} = J(\hat{f}, \hat{f}). \tag{32.3}$$

Here $\hat{\mathscr{D}}$ is the differential operator in dimensionless form, $\hat{f}$ denotes the non-dimensionalized distribution function, and

$$\xi \equiv (v_0/l\nu_0) = (\lambda/l), \tag{32.4}$$

where λ is the mean free path.

The parameter ξ, which is inversely proportional to the density, provides a measure of departures from local translational equilibrium. Clearly as

$\xi \to 0$ the collision term dominates, and we recover the Maxwellian velocity distribution. Furthermore, when $\xi \ll 1$, we may expect f to depart only slightly from f_0. This suggests that we use ξ as an expansion parameter and assume that f can be written in the form

$$f = f_0 + f_1 + f_2 + \ldots = f_0(1 + \Phi_1 + \Phi_2 + \ldots) \qquad (32.5)$$

where each successive term is of progressively higher order in ξ so that

$$\hat{f} = \hat{f}_0 + \xi\hat{f}_1 + \xi^2\hat{f}_2 + \ldots = \hat{f}_0(1 + \xi\hat{\Phi}_1 + \xi^2\hat{\Phi}_2 + \ldots). \qquad (32.6)$$

In principle, this procedure provides a systematic expansion of f in powers of (λ/l). In practice, because we know that $\xi \ll 1$ for flows of interest (cf. §29), the first-order solution is a very good approximation.

We now substitute (32.6) into (32.3) and sort terms into groups of equal order in the parameter ξ. Reverting to dimensional variables we obtain a hierarchy of equations of the form

$$J(f_0, f_0) = 0, \qquad (32.7)$$

$$\mathcal{D}f_0 = J(f_0, f_1) + J(f_1, f_0), \qquad (32.8)$$

$$\mathcal{D}f_1 = J(f_0, f_2) + J(f_1, f_1) + J(f_2, f_0), \qquad (32.9)$$

and so on.

The zero-order equation (32.7) is the familiar one that yields the Maxwellian velocity distribution f_0. Given f_0, the first-order equation (32.8) provides an integral equation for f_1; it is this equation that we solve in what follows. Likewise, given f_0 and f_1, (32.9) provides an integral equation for f_2, and so on.

THE ZERO-ORDER EQUATION

We have already seen in §§8 and 9 that the solution of

$$J(f_0, f_0) = \iint [f_0(\mathbf{u}')f_0(\mathbf{u}_1') - f_0(\mathbf{u})f_0(\mathbf{u}_1)]g\sigma(\mathbf{\Omega})\, d\Omega\, d^3u_1 = 0 \qquad (32.10)$$

is

$$f_0(\mathbf{u}) = N(m/2\pi kT)^{3/2} \exp[-m(\mathbf{u}-\mathbf{v})^2/2kT] \qquad (32.11)$$

where $\mathbf{v} = \langle \mathbf{u} \rangle$. In the present context, we demand only that f_0 satisfy (32.10) locally, and do not impose the further requirement that the material be in strict equilibrium. This means that, because (32.10) by itself places no restrictions on the dependence of N, T, and $\mathbf{v}$ on $\mathbf{x}$ and t, we may regard

$$N = N(\mathbf{x}, t); \qquad T = T(\mathbf{x}, t); \qquad \text{and} \qquad \mathbf{v} = \mathbf{v}(\mathbf{x}, t) \qquad (32.12)$$

as arbitrary functions of $\mathbf{x}$ and t.

From the functional form of f_0 one finds the purely mathematical results that

$$\int f_0 \, d^3u = N, \tag{32.13}$$

$$\int \mathbf{u} f_0 \, d^3u = N\mathbf{v}, \tag{32.14}$$

and

$$\int \tfrac{1}{2} m(\mathbf{u}-\mathbf{v})^2 f_0 \, d^3u = \tfrac{3}{2} NkT. \tag{32.15}$$

But, at the same time, we wish to interpret N, T, and $\mathbf{v}$ physically as the density, temperature, and local flow velocity of the gas, which from (6.1), (6.2), and (30.18) are

$$\int f \, d^3u = N, \tag{32.16}$$

$$\int \mathbf{u} f \, d^3u = N\mathbf{v}, \tag{32.17}$$

and

$$\int \tfrac{1}{2} m(\mathbf{u}-\mathbf{v})^2 f \, d^3u = \tfrac{3}{2} NkT, \tag{32.18}$$

where f is now the actual distribution function. For these two sets of equations to be compatible, we evidently must require that, for all terms f_i with $i>0$ in (32.5),

$$\int f_i \, d^3u = 0, \tag{32.19}$$

$$\int \mathbf{u} f_i \, d^3u = 0, \tag{32.20}$$

and

$$\int U^2 f_i \, d^3u = 0, \tag{32.21}$$

where, as before, $\mathbf{U} \equiv \mathbf{u}-\mathbf{v}$. Equations (32.19) to (32.21) place important constraints on the allowable functional form of f.

THE FIRST-ORDER SOLUTION

In (32.5) we wrote the first-order term f_1 in the expansion of f as $f_1 = f_0 \Phi_1$, where according to (32.8), Φ_1 is a scalar function that satisfies the integral equation

$$\begin{aligned} -N^2 \mathscr{I}(\Phi_1) \equiv \iint & [\Phi_1(\mathbf{u}') + \Phi_1(\mathbf{u}_1') - \Phi_1(\mathbf{u}) - \Phi_1(\mathbf{u}_1)] \\ & \times f_0(\mathbf{u}) f_0(\mathbf{u}_1) g\sigma(\mathbf{\Omega}) \, d\Omega \, d^3u_1 = \mathscr{D} f_0. \end{aligned} \tag{32.22}$$

Here we have used $f_0(\mathbf{u}') f_0(\mathbf{u}_1') \equiv f_0(\mathbf{u}) f_0(\mathbf{u}_1)$ from (32.10).

To evaluate the right-hand side of (32.22), we recall that f_0 depends on $\mathbf{x}$ and t through the functions (32.12). We therefore expand $\mathscr{D}f_0$ in terms of derivatives with respect to N, T, and $\mathbf{v}$, and derivatives of these quantities with respect to $\mathbf{x}$ and t. We then eliminate time derivatives of N, T, and $\mathbf{v}$ by means of the conservation equations for equilibrium flow (given in §31), which are the relations appropriate to $f=f_0$.

We notice first that

$$\begin{aligned}\mathscr{D}f_0 &= (\partial f_0/\partial t)+u^i(\partial f_0/\partial x^i)+a^i(\partial f_0/\partial u^i)\\ &= (Df_0/Dt)+U^i(\partial f_0/\partial x^i)+a^i(\partial f_0/\partial u^i).\end{aligned} \tag{32.23}$$

Then, taking $f_0=f_0[N(\mathbf{x},t),T(\mathbf{x},t),\mathbf{v}(\mathbf{x},t)]$ we have

$$\begin{aligned}\mathscr{D}f_0 &= \frac{\partial f_0}{\partial N}\frac{DN}{Dt}+\frac{\partial f_0}{\partial v^i}\frac{Dv^i}{Dt}+\frac{\partial f_0}{\partial T}\frac{DT}{Dt}\\ &\quad +U^i\left(\frac{\partial f_0}{\partial N}\frac{\partial N}{\partial x^i}+\frac{\partial f_0}{\partial v^j}\frac{\partial v^j}{\partial x^i}+\frac{\partial f_0}{\partial T}\frac{\partial T}{\partial x^i}\right)+a^i\frac{\partial f_0}{\partial u^i}.\end{aligned} \tag{32.24}$$

But, from (32.11), one immediately finds

$$(\partial f_0/\partial N)=f_0/N, \tag{32.25}$$

$$(\partial f_0/\partial v^i)=(m/kT)(u^i-v^i)f_0=(mU^i/kT)f_0, \tag{32.26}$$

$$(\partial f_0/\partial u^i)=-(mU^i/kT)f_0 \tag{32.27}$$

and

$$(\partial f_0/\partial T)=[(mU^2/2kT)-\tfrac{3}{2}](f_0/T). \tag{32.28}$$

Furthermore, from (31.3) to (31.6),

$$(DN/Dt)=-Nv^i_{,i} \tag{32.29}$$

$$(Dv_i/Dt)=a_i-(p_{,i}/\rho), \tag{32.30}$$

and

$$(De/Dt)=(3k/2m)(DT/Dt)=-(p/\rho)v^i_{,i}. \tag{32.31}$$

On substituting (32.25) to (32.31) into (32.24), and using the equation of state $p=NkT$, we obtain, after some reduction,

$$\mathscr{D}f_0=\{U_i[(mU^2/2kT)-\tfrac{5}{2}](\ln T)_{,i}+(m/kT)(U_iU_j-\tfrac{1}{3}U^2\,\delta_{ij})v_{i,j}\}f_0. \tag{32.32}$$

We can cast (32.32) into a more useful form by noticing that, because the last term is symmetric in i and j, we can replace $v_{i,j}$ by $\frac{1}{2}(v_{i,j}+v_{j,i})$, and furthermore, because $(U_iU_j-\frac{1}{3}U^2\,\delta_{ij})$ is obviously traceless, we can add to $\frac{1}{2}(v_{i,j}+v_{j,i})$ a scalar times δ_{ij}; in particular we can add $-(\frac{1}{3}v_{k,k})\,\delta_{ij}$. Thus we can also write

$$\begin{aligned}\mathscr{D}f_0 &= \{U_i[(mU^2/2kT)-\tfrac{5}{2}](\ln T)_{,i}+(m/kT)(U_iU_j-\tfrac{1}{3}U^2\,\delta_{ij})D_{ij}\}f_0\\ &= -N^2\mathscr{I}(\Phi_1),\end{aligned} \tag{32.33}$$

where

$$D_{ij}\equiv\tfrac{1}{2}(v_{i,j}+v_{j,i})-(\tfrac{1}{3}v_{k,k})\,\delta_{ij}. \tag{32.34}$$

The reason for introducing the traceless tensor D_{ij} here will become clear shortly.

We are now in a position to deduce the form of Φ_1. Because Φ_1 is a scalar, and because $\mathcal{I}(\Phi_1)$ is linear in Φ_1, we conclude that there must be a particular solution for Φ_1 which is a linear combination of the components of ∇T and of D_{ij}. Moreover, because f_0 is proportional to N, it follows from (32.22) and (32.33) that Φ_1 must be inversely proportional to N. Finally, we notice that in (32.33) the coefficients of $T_{,i}$ and D_{ij} depend only on the temperature T and the particle random velocities U_i.

We therefore can conclude that there must be a particular solution for Φ_1 of the form

$$\Phi_1 = -N^{-1}[(2kT/m)^{1/2}A_i(\ln T)_{,i} + B_{ij}D_{ij} + \Psi] \tag{32.35}$$

where A_i, B_{ij}, and Ψ are unknown functions of T and U_i. The minus sign and the factor multiplying A_i have been chosen for later convenience. When (32.35) is substituted into (32.33) we find that A_i, B_{ij}, and Ψ must satisfy the integral equations

$$N\mathcal{I}(A_i) = \mathcal{U}_i(\mathcal{U}^2 - \tfrac{5}{2})f_0, \tag{32.36}$$

$$N\mathcal{I}(B_{ij}) = 2(\mathcal{U}_i\mathcal{U}_j - \tfrac{1}{3}\mathcal{U}^2\,\delta_{ij})f_0, \tag{32.37}$$

and

$$\mathcal{I}(\Psi) = 0, \tag{32.38}$$

where, for brevity, we have introduced the dimensionless velocity variable

$$\mathcal{U} = (m/2kT)^{1/2}\mathbf{U}. \tag{32.39}$$

The central problem of the Chapman–Enskog method is to find solutions for equations (32.36) to (32.38). We will carry out this solution in §33, but we can make considerable further progress if we deduce a partial solution for A_i and B_{ij} from the following simple observations. First, $\mathbf{A}$ is a vector which is a function of T and of the components of $\mathcal{U}$. The only vector that can be constructed from these quantities is $\mathcal{U}$ itself, multiplied by some scalar function of the scalars $\mathcal{U}$ and T. Hence we must have

$$A_i = A(\mathcal{U}, T)\mathcal{U}_i. \tag{32.40}$$

Next, (32.37) shows that $\mathcal{I}(B_{ij} - B_{ji}) = \mathcal{I}(B_{ij}) - \mathcal{I}(B_{ji}) \equiv 0$; hence B_{ij} must be symmetric. Furthermore, (32.37) implies that $\mathcal{I}(B_{xx} + B_{yy} + B_{zz}) \equiv 0$; hence B_{ij} must be traceless. In short, B_{ij} is a traceless tensor function of T and of the components of $\mathcal{U}$. But the only second-rank, symmetric, traceless tensor that can be constructed from these quantities is $(\mathcal{U}_i\mathcal{U}_j - \frac{1}{3}\mathcal{U}^2\,\delta_{ij})$ times a scalar function of the scalars $\mathcal{U}$ and T. Hence we can write

$$B_{ij} = B(\mathcal{U}, T)(\mathcal{U}_i\mathcal{U}_j - \tfrac{1}{3}\mathcal{U}^2\,\delta_{ij}). \tag{32.41}$$

Equations (32.40) and (32.41) provide only partial solutions for A_i and B_{ij}. In contrast, for the function Ψ we can obtain a complete solution as follows. Equation (32.28) is satisfied if Ψ is chosen to be a linear combination of the summational invariants:

$$\Psi = \alpha_1 + \boldsymbol{\alpha}_2 \cdot \mathbf{U} + \alpha_3 U^2, \tag{32.42}$$

where α_1 and α_3 are scalars and $\boldsymbol{\alpha}_2$ is a vector. In fact, it can be shown that this is the only possible form for Ψ (**C2,** §4.41 and p. 50). Now, from (32.19) to (32.21), Φ_1 must satisfy the three conditions

$$\int f_0 \Phi_1 \, d^3U = 0, \tag{32.43}$$

$$\int U_i f_0 \Phi_1 \, d^3U = 0, \tag{32.44}$$

and

$$\int U^2 f_0 \Phi_1 \, d^3U = 0. \tag{32.45}$$

Using (32.35) and (32.40) to (32.42) in (32.43) to (32.45), and retaining only nonzero integrals, we have

$$\int (\alpha_1 + \alpha_3 U^2) f_0 \, d^3U = 0, \tag{32.46}$$

$$\int [\alpha_{2i} + A(\mathcal{U}, T)(\ln T)_{,i}] U^2 f_0 \, d^3U = 0, \tag{32.47}$$

and

$$\int (\alpha_1 + \alpha_3 U^2) U^2 f_0 \, d^3U = 0. \tag{32.48}$$

Then, using the explicit form for f_0, one finds that (32.46) and (32.48) imply that $\alpha_1 + \alpha_3 \langle U^2 \rangle = 0$ and $\alpha_1 + 3\alpha_3 \langle U^2 \rangle = 0$, which can be satisfied only if $\alpha_1 = \alpha_3 = 0$. Equation (32.47) implies that the vector $\boldsymbol{\alpha}_2$ is parallel to ∇T. Hence, in view of (32.40), the term $\boldsymbol{\alpha}_2 \cdot \mathbf{U}$ in Ψ can be absorbed into the term $\mathbf{A} \cdot \nabla T$ in Φ_1 merely by modifying the value of $A(\mathcal{U}, T)$, which is as yet undetermined. Therefore in (32.35) we may simply set $\Psi \equiv 0$.

Finally, note in passing that (32.47) implies that

$$\int A(\mathcal{U}, T) \mathcal{U}^2 f_0 \, d^3\mathcal{U} = 0, \tag{32.49}$$

and

$$\int A(\mathcal{U}, T) \mathcal{U}_i^2 f_0 \, d^3\mathcal{U} = 0, \tag{32.50}$$

results that will prove useful later.

THE HEAT FLUX VECTOR

Now that we have an analytical form for f_1, we can evaluate the dominant term in the heat flux vector $\mathbf{q}$. If we use $f = f_0 + f_1$ in (30.26), and recall from §31 that $\mathbf{q}_0 \equiv 0$ when $f = f_0$, we find that

$$q_k = \tfrac{1}{2}m \int U_k U^2 f_0 \Phi_1 \, d^3U. \tag{32.51}$$

If we now substitute (32.35) into (32.51), we find that only a few terms survive the integration. In particular, the coefficient of D_{ij} contains integrals of the form $\int U_i U_j U_k U^2 B f_0 \, d^3U$ and $\int U_k U^4 B f_0 \, d^3U$. When we recall that B is a function only of the magnitude of U, and hence is an even function of the velocity components, it is clear that these integrals both vanish because the integrands are odd functions of the velocity components.

We therefore need consider only the term in ∇T. First, convert to dimensionless velocities via (32.39), and define

$$\mathscr{f}_0(\mathscr{U}) \equiv \pi^{-3/2} e^{-\mathscr{U}^2}, \tag{32.52}$$

while noting that $(m/2kT)^{3/2} \, d^3U = d^3\mathscr{U}$. Then

$$\begin{aligned} q_k &= \tfrac{1}{2}Nm[-N^{-1}(\ln T)_{,j}](2kT/m)^2 \int \mathscr{U}_k \mathscr{U}_j \mathscr{U}^2 A(\mathscr{U}, T) \mathscr{f}_0(\mathscr{U}) \, d^3\mathscr{U} \\ &= -(2k^2T/m) T_{,j} \, \delta_{jk} \int \mathscr{U}_l^2 \mathscr{U}^2 A(\mathscr{U}, T) \mathscr{f}_0(\mathscr{U}) \, d^3\mathscr{U} \\ &= -\left[(2k^2T/m) \int \mathscr{U}_l^2 \mathscr{U}^2 A(\mathscr{U}, T) \mathscr{f}_0(\mathscr{U}) \, d^3\mathscr{U}\right] T_{,k}. \end{aligned} \tag{32.53}$$

Comparing (32.53) with Fourier's law (27.18), we see that the factor in the square brackets is to be identified with the coefficient of thermal conductivity K.

The integral in (32.53) can be partly evaluated by integrating over angles. In this equation $\mathscr{U}_l$ denotes any single component of $\mathscr{U}$. For convenience, choose coordinate axes such that we can write $\mathscr{U}_l = \mathscr{U} \cos \theta$, and express $d^3\mathscr{U}$ as $d^3\mathscr{U} = \mathscr{U}^2 \, d\mathscr{U} \, d\omega = \mathscr{U}^2 \, d\mathscr{U} \, (\sin \theta \, d\theta \, d\phi)$, where $d\omega$ is an element of solid angle. The integrals over angles are trivial and we find that

$$K = (8\pi k^2 T/3m) \int A(\mathscr{U}, T) \mathscr{U}^6 \mathscr{f}_0(\mathscr{U}) \, d\mathscr{U}. \tag{32.54}$$

To complete the calculation of K we must determine the function $A(\mathscr{U}, T)$; we return to that problem in §33.

THE VISCOUS STRESS TENSOR

To evaluate the viscous stress tensor σ_{ij} we use $f = f_0 + f_1$ in (30.20). We already know from §31 that f_0 contributes nothing to σ_{ij}, and furthermore, from (32.21) we see that the second term in (30.20) vanishes identically for

$f=f_1$. Thus the only nonzero term in σ_{ij} is

$$\sigma_{ij}=-m\int U_iU_jf_0\Phi_1\,d^3U. \tag{32.55}$$

If we substitute (32.35) into (32.55), we again find that only a few terms survive the integration. In particular, the coefficient of $(\ln T)_{,k}$ contains an integral of the form $\int A(\mathcal{U},T)U_iU_jU_kf_0\,d^3U$, which is always zero because the integrand is an odd function of the velocity components. Thus only the terms in D_{ij} contribute to σ_{ij}, and we find

$$\begin{aligned}\sigma_{ij}&=(-Nm)(-N^{-1}D_{kl})(2kT/m)\int \mathcal{U}_i\mathcal{U}_jB_{kl}f_0(\mathcal{U})\,d^3\mathcal{U}\\&=2kTD_{kl}\int \mathcal{U}_i\mathcal{U}_j(\mathcal{U}_k\mathcal{U}_l-\tfrac{1}{3}\mathcal{U}^2\,\delta_{kl})B(\mathcal{U},T)f_0(\mathcal{U})\,d^3\mathcal{U}.\end{aligned} \tag{32.56}$$

Only the first of the two integrals on the right-hand side of (32.56) survives because D_{ij} is traceless, which implies that $D_{kl}\,\delta_{kl}\equiv 0$. Suppose now that $i\neq j$. Then the only integrals that do not vanish are those in which $k=i$ and $l=j$, or $k=j$ and $l=i$. Recalling that D_{ij} is symmetric, we therefore have

$$\begin{aligned}\sigma_{ij}&=\left[2kT\int \mathcal{U}_i^2\mathcal{U}_j^2B(\mathcal{U},T)f_0(\mathcal{U})\,d^3\mathcal{U}\right](2D_{ij})\\&=\left[2kT\int \mathcal{U}_i^2\mathcal{U}_j^2B(\mathcal{U},T)f_0(\mathcal{U})\,d^3\mathcal{U}\right](v_{i,j}+v_{j,i}-\tfrac{2}{3}v_{k,k}\,\delta_{ij}).\end{aligned} \tag{32.57}$$

By a slightly more complicated calculation, one can show that (32.57) remains valid when $i=j$. Comparing (32.57) with the viscous stress tensor for a Newtonian fluid as given by (25.3), we see that (1) the factor in the square brackets is to be identified with the coefficient of shear viscosity μ, and (2) kinetic theory predicts that the bulk viscosity is identically zero for a gas of monatomic structureless particles, a result mentioned earlier.

We can partially evaluate the integral in (32.57) by integrating over angles. Choose coordinates such that $\mathcal{U}_i=\mathcal{U}\cos\theta$ and $\mathcal{U}_j=\mathcal{U}\sin\theta\cos\phi$, and again write $d^3\mathcal{U}=\mathcal{U}^2\,d\mathcal{U}\,(\sin\theta\,d\theta\,d\phi)$. The angular integrals are straightforward and we obtain finally

$$\mu=(8\pi kT/15)\int B(\mathcal{U},T)\mathcal{U}^6f_0(\mathcal{U})\,d\mathcal{U}. \tag{32.58}$$

As was true for the thermal conductivity K, we must determine $B(\mathcal{U},T)$ in order to complete the calculation of μ; again we defer this problem until §33.

LIMITATIONS OF THE CHAPMAN-ENSKOG METHOD

While the Chapman–Enskog method is very powerful and provides transport coefficients that are in good agreement with experiment for a wide variety of gases and gas mixtures [cf. (**C2,** Chaps. 12–14)], it does nevertheless have limitations that should be borne in mind (**H2,** 18–21).

First, it is a classical theory, and hence fails in regimes where quantum effects dominate. For example, at low temperatures the de Broglie wavelength becomes large, and we can no longer consider the particles to be classical point particles moving on well-defined orbits. In addition, at high densities, quantum-statistical effects such as the Pauli exclusion principle come into operation, and the theory must be modified. Neither of these problems arises in the astrophysical media of interest to us, in which densities are relatively low and temperatures are high.

Second, the theory accounts only for binary collisions. Its validity can therefore be questioned in the important case of Coulomb collisions in an ionized gas for the reasons given in §10, where we saw that in a plasma, multiple, overlapping, weak collisions play the dominant role. However, as we will see in §33, it turns out that if the calculation is carried out to a sufficiently high degree of approximation, the Chapman–Enskog theory yields accurate results even in a plasma.

Third, the Chapman–Enskog method provides only a series solution to the Boltzmann equation, and there are questions about the convergence of the series. It is clear that the first-order approximation, the only one we discuss, can be expected to be valid only when $(\lambda/l) \ll 1$ and when gradients of physical properties are small. While these restrictions are usually met in the flows of interest to us, there are some situations (e.g., shocks) where gradients are so steep that the validity of both the Navier-Stokes equations and the Chapman–Enskog method becomes doubtful. In practice, this problem will not be a serious drawback for our work because we can rarely (if ever) observationally resolve shock structures in astrophysical flows; moreover, in shock computations the details of the shock structure are often purposely smeared out by use of an artificial viscosity in order to gain numerical stability. In contrast, for radiation, which can be considered to be a gas composed of photons, the condition $(\lambda/l) \ll 1$ will be met only in the interior of a star, whereas in the atmosphere $(\lambda/l) \gtrsim 1$, and photons escape freely into space. In the former case, one can write analogues to the Chapman–Enskog solution (cf. §§80 and 97); but in the latter case such an approach is worthless even in principle, and one must confront the full equation of transfer.

Fourth, the theory discussed above applies only to monatomic gases with no internal degrees of freedom. The effects of internal excitation and of ionization are thus ignored; we discuss these effects briefly below.

EFFECTS OF EXCITATION AND IONIZATION

The atoms in a gas have internal excitation states and can also ionize; molecules have, in addition, rotational and vibrational energy states. The existence of these extra energy states changes not only local thermodynamic properties (e.g., pressure or specific internal energy) of a gas, but also affects the transport of momentum and energy in nonuniform media. Thus in analyzing the flow of an excited and ionized fluid we must be careful to use the correct constitutive relations for quantities such as p and e, and also the correct transport coefficients. In what follows, we assume the material is in LTE (local thermodynamic equilibrium).

The pressure in a gas results from exchange of momentum associated with translational motions and hence is unaffected by the existence of internal excitation states. Ionization, in contrast, contributes extra particles and therefore alters the equation of state $p(\rho, T)$ (cf. §14).

The specific internal energy of a gas is affected to first order by the availability of nontranslational energy states. Thus, an excited and ionizing monatomic gas will have $e = e_{\text{trans}} + e_{\text{exc}} + e_{\text{ion}}$ [see equation (14.11)], and in a gas that includes molecules one must allow for yet other terms in e_{vib}, e_{rot}, and e_{dissoc}. Similar modifications can be made for other thermodynamic properties such as specific heats.

In the calculation of transport coefficients, internal excitation and ionization can contribute significantly to the transport of energy. For example when we calculate the heat flux vector in a gas we should account for both the translational kinetic energy and the internal excitation energy transported by the translational motions of the particles. Thus, in place of (30.26), we should write

$$\mathbf{q} = \rho\langle\tfrac{1}{2}U^2\mathbf{U}\rangle + \rho\langle e_{\text{exc}}\mathbf{U}\rangle. \tag{32.59}$$

The second term should, in principle, include all forms of nontranslational energy available to the particles (electronic, rotational, vibrational, ionization, dissociation). The evaluation of this term is complicated by the need to specify how rapidly the internal energy transported by a particle is communicated to other particles in its surroundings, which requires large amounts of atomic cross-section data to describe the various kinds of interactions that can occur. In the absence of such data, resort has been made to some rather artificial models [cf. (**C2**, Chap. 11)], and an approximate result known as the *Eucken correction* has often been used (**H2**, §7.66). We cite more accurate detailed work for hydrogen-helium mixtures in §33. Another major effect of ionization in a strongly ionized gas is to change the interparticle interaction from a short-range potential, appropriate for neutral particles, to the long-range Coulomb interaction; as we shall see in §33, this transition radically alters the temperature dependence of both K and μ. As was mentioned earlier in §25, detailed analysis also shows that the finite time required to exchange energy between translational and internal excitation modes results in a nonzero bulk viscosity (**C2**, §11.51), (**H2**, §7.6c), (**V1**, §10.8).

33. Evaluation of the Transport Coefficients

In order to calculate the thermal conductivity and the shear viscosity from equations (32.54) and (32.58), we must solve the integral equations (32.36) and (32.37) for the scalar functions $A(\mathcal{U}, T)$ and $B(\mathcal{U}, T)$. The standard approach used to obtain these solutions is to expand A and B in a series of special polynomials, the *Sonine polynomials*, which are orthogonal for weighting functions of the form $x^n e^{-x^2}$. Chapman and Cowling use an infinite series of these polynomials and express the transport coefficients as ratios of infinite determinants (**C2,** §§7.51 and 7.52). Their results have practical value because one finds that the ratio of the determinants converges very rapidly as rows and columns are added, so that it is actually necessary to consider only a few terms in the expansion. Alternatively, A and B can be expanded in Sonine polynomials and the integral equations solved by application of a variational principle (**H2,** §§7.3 and 7.4). In their full generality, both methods are quite complicated, and it would take us too far afield to discuss them here. Instead we apply the variational principle to the simplest possible trial functions for A and B, which, despite their simplicity, yield precisely the same first-order results as are obtained from the other methods [cf. (**B2**, §§7.9 and 7.10)].

THE VARIATIONAL PRINCIPLE

We show below that the expressions for both K and μ can be cast into the general form

$$H = C\left[2\int FG\, d^3\mathcal{U} - \int F\mathcal{I}(F)\, d^3\mathcal{U}\right]. \tag{33.1}$$

Here H denotes either K or μ, and the function F is the solution of the integral equation

$$\mathcal{I}(F) = G, \tag{33.2}$$

where $\mathcal{I}$ is the Boltzmann collision operator (32.22).

Consider now how the value of H will change for small variations δF of F around its correct value as given by the exact solution of (33.2). Writing out $K \equiv \int F\mathcal{I}(F)\, d^3\mathcal{U}$ in a form similar to equation (7.42), it is easy to show that $\delta K = 2\int \delta F\mathcal{I}(F)\, d^3\mathcal{U}$, hence

$$\delta H = 2C\int \delta F\,[G - \mathcal{I}(F)]\, d^3\mathcal{U}. \tag{33.3}$$

If F is, in fact, the solution of (33.2), then $\delta H = 0$. That is, H is *stationary* with respect to small variations of F around its true form; therefore it is reasonable to expect that even an approximate expression for F will yield a good estimate of H.

In particular, suppose we choose a trial function F_0, and write $F = \alpha F_0$, where α is a variational parameter to be determined in such a way as to optimize the value of H. The optimum choice of α is that for which H is stationary with respect to variations in α. Thus, starting with

$$H(\alpha) = C\left[2\alpha\int F_0 G\, d^3\mathcal{U} - \alpha^2\int F_0\mathcal{I}(F_0)\, d^3\mathcal{U}\right], \tag{33.4}$$

we demand that

$$\delta H(\alpha) = 2C\left[\int F_0 G\, d^3\mathcal{U} - \alpha \int F_0 \mathcal{I}(F_0)\, d^3\mathcal{U}\right] \delta\alpha \equiv 0. \tag{33.5}$$

which yields

$$\alpha = \left(\int F_0 G\, d^3\mathcal{U}\right) \Big/ \left(\int F_0 \mathcal{I}(F_0)\, d^3\mathcal{U}\right), \tag{33.6}$$

whence the optimum estimate of H is

$$H = \frac{C\left(\int F_0 G\, d^3\mathcal{U}\right)^2}{\int F_0 \mathcal{I}(F_0)\, d^3\mathcal{U}}. \tag{33.7}$$

We must now show that the expressions for K and μ given in §32 can actually be cast into the form of (33.1). Consider first K. From (32.49) we have

$$-\tfrac{5}{2}\int A(\mathcal{U}, T)\mathcal{U}^2 f_0(\mathcal{U})\, d^3\mathcal{U} \equiv 0, \tag{33.8}$$

thus (32.54) can be rewritten as

$$K = (2k^2T/3m)\int A(\mathcal{U}, T)\mathcal{U}^2(\mathcal{U}^2 - \tfrac{5}{2}) f_0(\mathcal{U})\, d^3\mathcal{U}. \tag{33.9}$$

Furthermore, by virtue of (32.40), $A(\mathcal{U}, T)\mathcal{U}^2 = A(\mathcal{U}, T)\mathcal{U}_i\mathcal{U}_i = A_i\mathcal{U}_i$; therefore

$$K = (2k^2T/3m)\int A_i\mathcal{U}_i(\mathcal{U}^2 - \tfrac{5}{2}) f_0(\mathcal{U})\, d^3\mathcal{U}. \tag{33.10}$$

But recall that A_i also satisfies (32.36); hence (33.10) is equivalent to

$$K = (2k^2T/3m)\int A_i \mathcal{I}(A_i)\, d^3\mathcal{U}. \tag{33.11}$$

Thus, combining (33.10) and (33.11), we have

$$K = (2k^2T/3m)\left[2\int A_i\mathcal{U}_i(\mathcal{U}^2 - \tfrac{5}{2}) f_0(\mathcal{U})\, d^3\mathcal{U} - \int A_i \mathcal{I}(A_i)\, d^3\mathcal{U}\right] \tag{33.12}$$

which is clearly of the form (33.1) with $F = A_i$ and $G = \mathcal{U}_i(\mathcal{U}^2 - \tfrac{5}{2}) f_0(\mathcal{U})$.

Consider now μ. We can cast (32.58) into a more useful form by defining the traceless tensor

$$\mathcal{U}_l^{\circ}\mathcal{U}_m \equiv \mathcal{U}_l\mathcal{U}_m - \tfrac{1}{3}\mathcal{U}^2\,\delta_{lm}, \tag{33.13}$$

and by noting the identity

$$(\mathcal{U}_l^{\circ}\mathcal{U}_m)(\mathcal{U}_l^{\circ}\mathcal{U}_m) = \tfrac{2}{3}\mathcal{U}^4, \tag{33.14}$$

which follows directly by substitution from (33.13) when we recall that $\delta_{lm}\,\delta_{lm}=3$. Then (32.58) becomes

$$\mu=\tfrac{1}{5}kT\int B(\mathcal{U},T)\,\mathcal{U}_l^\circ\mathcal{U}_m\mathcal{U}_l^\circ\mathcal{U}_m f_0(\mathcal{U})\,d^3\mathcal{U} \tag{33.15a}$$

$$=\tfrac{1}{10}kT\int B_{lm}\,2\mathcal{U}_l^\circ\mathcal{U}_m f_0(\mathcal{U})\,d^3\mathcal{U} \tag{33.15b}$$

$$=\tfrac{1}{10}kT\int B_{lm}\mathscr{I}(B_{lm})\,d^3\mathcal{U}, \tag{33.15c}$$

where we have used (32.41) and (32.37). Combining (33.15b) and (33.15c) we have

$$\mu=\tfrac{1}{10}kT\left[2\int B_{lm}\,2\mathcal{U}_l^\circ\mathcal{U}_m f_0(\mathcal{U})\,d^3\mathcal{U}-\int B_{lm}\mathscr{I}(B_{lm})\,d^3\mathcal{U}\right], \tag{33.16}$$

which again is clearly of the form (33.1) with $F=B_{lm}$ and $G=2\mathcal{U}_l^\circ\mathcal{U}_m f_0(\mathcal{U})$.

CALCULATION OF K AND μ

In order to apply (33.7) to (33.12) and (33.16) we must make an appropriate choice of the trial function F_0 in each case. In view of (32.40), to calculate K we write

$$F=\alpha F_0=\alpha A_i^0=\alpha A^0\mathcal{U}_i \tag{33.17}$$

where α is the variational parameter and A^0 is a trial function for $A(\mathcal{U},T)$. Similarly, to calculate μ we choose, bearing in mind (32.41),

$$F=\alpha B_{lm}^0=\alpha B^0\mathcal{U}_l^\circ\mathcal{U}_m, \tag{33.18}$$

where B^0 is a trial function for $B(\mathcal{U},T)$.

Both A^0 and B^0 must be chosen such that Φ_1 will still satisfy the restrictions imposed by (32.43) to (32.45) if we use our guesses for A_i and B_{ij} in (32.35); in fact we can impose these requirements on each trial function separately. If we do this, it is easy to see that the simplest choice of B^0 that meets the requirements is $B^0=\text{constant}$. In this case we can absorb the constant into α, and set $B^0\equiv 1$ in (33.18). The situation for A^0 is more complicated. The choice $A^0=\text{constant}$ is unsuccessful because it fails to satisfy (32.49), which is a consequence of (32.43) to (32.45). Comparing (32.36) and (32.40) leads one to suspect that $A(\mathcal{U},T)$ should be expanded in even powers of $\mathcal{U}$, a conjecture shown to be valid by the Sonine polynomial analysis [cf. (**C2,** §7.51)]. Hence we try $A^0=a+b\mathcal{U}^2$, noting that one of the constants can be absorbed into the variational parameter, so that only their ratio matters. Substituting this trial solution into (32.49), one finds that $(a/b)=-\tfrac{5}{2}$; hence in (33.17) we set $A^0=(\mathcal{U}^2-\tfrac{5}{2})$.

Using these trial functions in (33.7) we find

$$[K]_1 = \left(\frac{2k^2T}{3m}\right)\frac{\left[\int(\mathcal{U}^2-\frac{5}{2})^2\mathcal{U}^2 f_0(\mathcal{U})\,d^3\mathcal{U}\right]^2}{\left[\int(\mathcal{U}^2-\frac{5}{2})\mathcal{U}_i\mathcal{I}[\mathcal{U}_i(\mathcal{U}^2-\frac{5}{2})\right]d^3\mathcal{U}} \equiv \left(\frac{2k^2T}{3m}\right)\frac{I_1^2}{I_2}, \tag{33.19}$$

and, recalling (33.14),

$$[\mu]_1 = \left(\frac{kT}{10}\right)\frac{\left[\frac{4}{3}\int\mathcal{U}^4 f_0(\mathcal{U})\,d^3\mathcal{U}\right]^2}{\int\mathcal{U}_l^{\circ}\mathcal{U}_m\mathcal{I}(\mathcal{U}_l^{\circ}\mathcal{U}_m)\,d^3\mathcal{U}} \equiv \left(\frac{kT}{10}\right)\frac{I_3^2}{I_4}. \tag{33.20}$$

The notation emphasizes that these expressions are only first approximations to K and μ.

The integrals I_1 and I_3 are straightforward, and with the help of standard tables, we find $I_1=\frac{15}{4}$ and $I_3=5$. The integrals I_2 and I_4 are more complicated. Consider first I_2. Introducing the dimensionless relative velocity $\boldsymbol{\gamma}\equiv(m/2kT)^{1/2}\mathbf{g}$, and using the definition of $\mathcal{I}$, we have

$$I_2 = (2kT/m)^{1/2}\int d^3\mathcal{U}\int d^3\mathcal{U}_1\oint\sigma(\boldsymbol{\Omega})\,d\Omega[\![f_0(\mathcal{U})f_0(\mathcal{U}_1)\gamma \times\{(\mathcal{U}^2-\tfrac{5}{2})\mathcal{U}\cdot[\mathcal{U}(\mathcal{U}^2-\tfrac{5}{2})+\mathcal{U}_1(\mathcal{U}_1^2-\tfrac{5}{2})-\mathcal{U}'(\mathcal{U}'^2-\tfrac{5}{2})-\mathcal{U}_1'(\mathcal{U}_1'^2-\tfrac{5}{2})]\}]\!] \tag{33.21}$$

From the results of §7 in dimensionless form for particles of equal masses, we can write

$$\mathcal{U}+\mathcal{U}_1=\mathcal{U}'+\mathcal{U}_1'=2\boldsymbol{\Gamma}, \tag{33.22}$$

$$\mathcal{U}^2+\mathcal{U}_1^2=\mathcal{U}'^2+\mathcal{U}_1'^2=2\Gamma^2+\tfrac{1}{2}\gamma^2, \tag{33.23}$$

$$\mathcal{U}=\boldsymbol{\Gamma}+\tfrac{1}{2}\boldsymbol{\gamma} \tag{33.24a}$$

$$\mathcal{U}'=\boldsymbol{\Gamma}+\tfrac{1}{2}\boldsymbol{\gamma}', \tag{33.24b}$$

and

$$\mathcal{U}_1=\boldsymbol{\Gamma}-\tfrac{1}{2}\boldsymbol{\gamma} \tag{33.24c}$$

$$\mathcal{U}_1'=\boldsymbol{\Gamma}-\tfrac{1}{2}\boldsymbol{\gamma}', \tag{33.24d}$$

where $\boldsymbol{\Gamma}$ is the dimensionless center of mass velocity. Then from (7.35) we have

$$d^3\mathcal{U}\,d^3\mathcal{U}_1=(4\pi\gamma^2\,d\gamma)(4\pi\Gamma^2\,d\Gamma)(d\omega_\gamma/4\pi)(d\omega_\Gamma/4\pi), \tag{33.25}$$

where $d\omega_\gamma$ and $d\omega_\Gamma$ are the solid angles describing the orientations of $\boldsymbol{\gamma}$ and $\boldsymbol{\Gamma}$. Furthermore,

$$f_0(\mathcal{U})f_0(\mathcal{U}_1)=\pi^{-3}\exp[-(\mathcal{U}^2+\mathcal{U}_1^2)]=\pi^{-3}\exp[-(2\Gamma^2+\tfrac{1}{2}\gamma^2)], \tag{33.26}$$

and

$$\mathcal{U}^2-\tfrac{5}{2}=(\Gamma^2+\tfrac{1}{4}\gamma^2-\tfrac{5}{2})+\boldsymbol{\gamma}\cdot\boldsymbol{\Gamma}. \tag{33.27}$$

From (33.22) one immediately sees that all terms containing the numerical factor $\frac{5}{2}$ in the square bracket of (33.21) cancel one another and can be dropped henceforth. Next, if we expand the remaining terms in the square bracket, we find

$$\begin{aligned}\boldsymbol{\mathcal{U}}\cdot[\mathcal{U}^2\boldsymbol{\mathcal{U}}+\mathcal{U}_1^2\boldsymbol{\mathcal{U}}_1-\mathcal{U}'^2\boldsymbol{\mathcal{U}}'-\mathcal{U}_1'^2\boldsymbol{\mathcal{U}}_1'] \\ =(\boldsymbol{\gamma}\cdot\boldsymbol{\Gamma})^2-(\boldsymbol{\gamma}'\cdot\boldsymbol{\Gamma})^2+\tfrac{1}{2}\gamma^2[\boldsymbol{\gamma}\cdot\boldsymbol{\Gamma}-(\boldsymbol{\gamma}'\cdot\boldsymbol{\Gamma})\cos\chi]\end{aligned} \tag{33.28}$$

where χ is the angle between $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$, as usual. The product of (33.26), (33.27), and (33.28) must be integrated over all $d\omega_\Gamma$ for fixed $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$. Notice that all the terms in the parenthesis of (33.27) are independent of the orientation of $\boldsymbol{\Gamma}$. When integrated over $(d\omega_\Gamma/4\pi)$, the product of these terms with those in (33.28) must average to zero, because if ψ and ψ' are the angles between $\boldsymbol{\Gamma}$ and $\boldsymbol{\gamma}$, and between $\boldsymbol{\Gamma}$ and $\boldsymbol{\gamma}'$, respectively, then the angle average of the first two terms is of the form $\gamma^2\Gamma(\langle\cos^2\psi\rangle-\langle\cos^2\psi'\rangle)\equiv 0$, and the averages of the last two terms contain the factors $\langle\cos\psi\rangle=\langle\cos\psi'\rangle\equiv 0$. Here we use the result from (7.8) that $\gamma'=\gamma$. Therefore, only the term in $\boldsymbol{\gamma}\cdot\boldsymbol{\Gamma}$ from (33.27) survives, and it, in turn, clearly can yield a nonzero average only with the two terms in the square bracket of (33.28).

Let us now decompose $\boldsymbol{\gamma}'$ into components along $\boldsymbol{\gamma}$ and along a unit vector $\mathbf{n}$ perpendicular to $\boldsymbol{\gamma}$; then

$$\boldsymbol{\gamma}'=(\cos\chi)\boldsymbol{\gamma}+\gamma(\sin\chi)\mathbf{n}, \tag{33.29}$$

and the terms to be integrated over $d\omega_\Gamma$ are $\frac{1}{2}\gamma^2[(\boldsymbol{\gamma}\cdot\boldsymbol{\Gamma})^2(1-\cos^2\chi)-\gamma\sin\chi\cos\chi(\boldsymbol{\gamma}\cdot\boldsymbol{\Gamma})(\mathbf{n}\cdot\boldsymbol{\Gamma})]$. Because $\boldsymbol{\gamma}$ and $\mathbf{n}$ are orthogonal, the last term must average to zero, and we thus need compute only the integral of $\frac{1}{2}\gamma^4\Gamma^2\cos^2\psi(1-\cos^2\chi)$. Hence (33.21) has been reduced to

$$\begin{aligned}I_2=\frac{1}{2\pi^3}\left(\frac{2kT}{m}\right)^{1/2}\int_0^\infty d\gamma(4\pi\gamma^7e^{-\gamma^2/2})\oint d\Omega\sigma(\gamma,\chi)\sin^2\chi \\ \times\int_0^\infty d\Gamma(4\pi\Gamma^4e^{-2\Gamma^2})\oint\frac{\cos^2\psi}{4\pi}\,d\omega_\Gamma,\end{aligned} \tag{33.30}$$

where we have omitted the trivial integral over $d\omega_\gamma$.

In (33.30), the fourth integral is trivial; the third integral is given in standard tables; and the second integral is $\sigma_{(2)}(\gamma)$ as defined by equation (7.24). Thus, if we rewrite the first integral in terms of a new variable $y^2\equiv\frac{1}{2}\gamma^2$, we obtain, finally,

$$I_2=4(kT/\pi m)^{1/2}\int_0^\infty y^7\sigma_{(2)}(y)e^{-y^2}\,dy \tag{33.31}$$

and hence

$$[K]_1=\tfrac{75}{32}(\pi k^3T/m)^{1/2}\Big/\int_0^\infty y^7\sigma_{(2)}(y)e^{-y^2}\,dy. \tag{33.32}$$

By an analysis similar to that outlined above it is not difficult to show that

$I_4 \equiv I_2$, and therefore that

$$[\mu]_1 = \tfrac{5}{8}(\pi mkT)^{1/2} \Big/ \int_0^\infty y^7 \sigma_{(2)}(y) e^{-y^2}\, dy. \tag{33.33}$$

From (33.32) and (33.33) we have amonatomic gas is

$$[K]_1 = \tfrac{15}{4}(k/m)[\mu]_1 = \tfrac{5}{2}c_v[\mu]_1 = \tfrac{3}{2}c_p[\mu]_1, \tag{33.34}$$

which shows that the Prandtl number for a monatomic gas is

$$\mathrm{Pr} = c_p\mu/K = \tfrac{2}{3} \tag{33.35}$$

as stated earlier in §29.

RESULTS FOR SIMPLE MOLECULAR MODELS

For a given choice of an interparticle potential we can determine $\sigma_{(2)}(\gamma)$ as described in §7, and hence evaluate $[K]_1$ and $[\mu]_1$ explicitly. The simplest case is for rigid spheres, for which $\sigma_{(2)}$ as given by (7.25) is a constant. Substituting this value into (33.33) we obtain

$$[\mu]_1 = (5/16d^2)(mkT/\pi)^{1/2}, \tag{33.36}$$

which differs from the result given by a mean-free-path analysis only by a factor of order unity ($5\pi/16 = 0.982$). Given $[\mu]_1$, $[K]_1$ can be computed from (33.34). Chapman and Cowling calculate the correction from the first to the fourth degree of approximation (which is essentially exact) and show that $[\mu]_4 = 1.01600[\mu]_1$, $[K]_4 = 1.02513[K]_1$, and $[K]_4 = 2.5225[\mu]_4 c_v$.

The next simplest model is for an inverse-power potential. Using (7.26) in (33.33), one finds

$$[\mu]_1 = \frac{5}{8}\left(\frac{mkT}{\pi}\right)^{1/2}\left(\frac{2kT}{\alpha C_\alpha}\right)^{2/\alpha} \Big/ \left[A_2(\alpha)\Gamma\left(4-\frac{2}{\alpha}\right)\right], \tag{33.37}$$

where Γ denotes the standard gamma function. The thermal conductivity $[K]_1$ is again given in terms of $[\mu]_1$ by (33.34). Both $[\mu]_1$ and $[K]_1$ are proportional to T^s where

$$s = \tfrac{1}{2} + (2/\alpha). \tag{33.38}$$

Rigid-sphere molecules represent the limit $\alpha \to \infty$, in which case $s \to \frac{1}{2}$, in agreement with (33.36). Chapman and Cowling (**C2,** 173) give formulae that correct (33.37) to the second approximation.

The models discussed above are manifestly crude (although they agree remarkably well with experiment for many gases). More realistic interaction potentials allow for a strong repulsive term at small particle separations and a weak attractive term at large separations. The calculation of transport coefficients for such potentials is rather more complicated than for the models considered above; the details are discussed extensively in (**C2,** §§10.4 and 10.5), (**H2,** §8.4), and the references cited therein.

IONIZED GASES

The results derived above apply to neutral gases with short-range potentials. For ionized gases, the interaction is described by the Coulomb potential, which has $\alpha = 1$. For this value of α we expect from (33.38) that both K and μ will vary as $T^{5/2}$, which is markedly different from the temperature dependence found in a neutral gas. This result can be understood physically by the following dimensional argument. From (29.23) we see that the thermal conductivity is proportional to $(\tilde{e}/T)N\lambda\langle U\rangle$ where $\tilde{e}$ is the energy per particle, N is the number density of particles, λ is the mean free path, and $\langle U\rangle$ is the average transport velocity. But $(\tilde{e}/T)$ is merely a numerical factor, while $N\lambda \sim N/(N\sigma) \sim 1/\sigma$ where σ is the collision cross section. Writing $\sigma \approx \pi d_c^2$ where d_c is a typical collision radius, we estimate d_c by equating a particle's kinetic energy to the potential energy of the interaction: $\frac{1}{2}mU^2 \approx e^2/d_c$, which implies that $d_c \propto U^{-2}$ and thus $\sigma \propto U^{-4}$. Hence $K \sim \langle U\rangle/\sigma \sim U^5$; but $U \sim T^{1/2}$, hence $K \sim T^{5/2}$. From (29.25), one sees that the viscosity coefficient will have the same temperature dependence.

Although (33.38) gives the correct temperature dependence of K and μ, (33.37) cannot be applied immediately because $A_2(1)$ calculated from (7.27) diverges. To prevent this divergence, one must introduce a cutoff b_{max} in (7.16) and (7.27), for the same reasons a cutoff was needed in the calculations of relaxation times in §10. Once again we set $b_{\text{max}} = D$, the Debye length [cf. equation (10.15)]; one then finds

$$A_2(1) = 2[\ln(1+x^2) - x^2/(1+x^2)] \tag{33.39}$$

where $x \equiv 4kTD/e^2$. With this value for A_2 we can then write

$$[\mu]_1 = \tfrac{5}{8}(mkT/\pi)^{1/2}(2kT/e^2)^2/A_2(1) \tag{33.40}$$

and

$$[K]_1 = (75/32)(k^3T/\pi m)^{1/2}(2kT/e^2)^2/A_2(1). \tag{33.41}$$

If we set $m = m_e$ or $m = m_p$ in (33.40) and (33.41), we obtain transport coefficients for a gas of pure electrons or protons respectively (even though such gases of course do not exist in nature). We then notice that $K_e = (m_p/m_e)^{1/2}K_p \approx 43K_p$, and $\mu_e = (m_e/m_p)^{1/2}\mu_p \approx \mu_p/43$, so that thermal conduction is dominantly by electrons, and viscous forces are dominated by protons. This is what one would expect intuitively because in equilibrium both electrons and protons have the same thermal kinetic energies, but the electron random velocities, and hence the flux of electrons, are larger than the proton velocities by a factor of $(m_p/m_e)^{1/2}$, which implies that the electron thermal energy flux is larger by that factor. In contrast, for viscosity it is momentum transport that matters, and here the protons dominate because their momenta in the macroscopic flow are a factor of (m_p/m_e) larger than the electrons' momenta. Thus even though the proton flux across a surface is a factor of $(m_e/m_p)^{1/2}$ smaller than the electron flux, protons still transport more momentum by a net factor of $(m_p/m_e)^{1/2}$.

To obtain useful results for a real hydrogen plasma one must consider a *binary gas mixture* of protons and electrons. The theory then becomes much more complicated, and lies too far afield to discuss here, so we will merely quote results; the interested reader can refer to (**C2,** Chaps. 8 and 9) or (**H2,** §8.2) for details. Chapman (**C1**) has given a concise discussion of transport coefficients for completely ionized gases. He derives expressions that give the ionic and electronic contributions to both μ and K in a form that displays explicitly how each contribution is affected by collisions of the ions and electrons with other ions and electrons. The final results for completely ionized pure hydrogen are that $[\mu/\mu_p]_1 = 0.96$ and $[K/K_e]_1 = 0.324$, where μ_p and K_e are given by (33.40) and (33.41). These results apply only in the first approximation; in higher approximations both the basic computation of μ_p, μ_e, K_p, and K_e, and the formulae for combining these into transport coefficients for the mixed gas become more complicated. However, because μ is dominated by μ_p, and K by K_e, it is reasonable to suppose that the next approximation to K and μ will behave in much the same way as the next approximation for K_e and μ_p. From the formulae given by Chapman and Cowling (**C2,** 173) one finds $[\mu_p]_2 = 1.15[\mu_p]_1$ and $[K_e]_2 = 1.25[K_e]_1$. Work by Landshoff (**L2**), (**L3**) shows that changes produced by yet higher approximations are quite small.

A rather different approach was taken by Spitzer and Härm (**S3**), who solved numerically the Fokker-Planck equation for a completely ionized hydrogen plasma; this approach accounts fully for all mutual interactions among the ions and electrons and provides an accurate nonequilibrium distribution function. Results are presented for the electrical and thermal conductivity of the plasma (but not its viscosity); these results are conveniently expressed in terms of the conductivity of a *Lorentz gas* (an hypothetical fully ionized gas in which the electrons do not interact with one another and the protons are at rest), which is given by

$$K_L = 40(2k^7T^5/\pi^3m)^{1/2}/e^4 \ln \Lambda \qquad (33.42)$$

where Λ is defined by (10.18). In an actual plasma $K = \delta_T K_L$; for hydrogen, Spitzer and Härm calculate $\delta_T = 0.225$. The results of the Fokker-Planck computation are found to be in good agreement with those from the Chapman-Enskog method provided that the latter are carried out to a sufficiently high degree of approximation.

A temperature gradient in a plasma alters the velocity distribution in such a way as to produce a heat flux transported by electrons; this net flow of electrons also implies that an electric current flows. But in a steady-state plasma, no current must flow in the direction of the temperature gradient because it would rapidly rise without limit. This paradox is resolved by recognizing that a secondary electric field is produced, which exactly cancels the current induced by the temperature gradient; at the same time it necessarily reduces the heat flow. To account for these *thermoelectric effects* the conductivity $K = \delta_T K_L$ calculated thus far must be reduced to an effective conductivity εK; for a pure hydrogen plasma Spitzer and Härm compute $\varepsilon = 0.419$. [See (**S2,** §5.5) or (**S3**) for further discussion.]

All of the results described above apply to completely ionized plasmas. For partially ionized gases, the case of interest in most stellar envelopes, the situation is yet more complex, and we can only cite results. Extensive calculations for μ and K for a partially ionized plasma of hydrogen and helium over a wide range of temperatures and pressures have been tabulated and discussed in detail by Edmonds (**E1**). Comprehensive calculations of the thermal conductivity in hydrogen-helium plasmas, using improved interaction cross sections and fourth-order Chapman–Enskog theory have been made by Ulmschneider (**U1**). The results are given in graphical form for a wide range of temperatures and pressures.

In conclusion it is worth remarking that energy transport by thermal conduction is usually negligible in comparison with radiative transport in most stellar atmospheres and envelopes [cf. Tables 2 and 3 of (**E1**)]. The important exception arises in stellar coronae where million-degree temperatures imply a high thermal conductivity, and these temperatures and low densities imply a high degree of ionization and hence low radiative efficiency. In the same vein, viscosity plays an important role in determining shock structure, but as was mentioned earlier, these structures cannot normally be resolved observationally or numerically. Even so, accurate viscosity coefficients prove very useful in a variety of applications, for example, estimates of wave damping.

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